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Citation for published version:

Li, C & Zhao, X 2019, 'Birational models of moduli spaces of coherent sheaves on the projective plane', *Geometry & Topology*, vol. 23, no. 1, pp. 347-426. <https://doi.org/10.2140/gt.2019.23.347>

Digital Object Identifier (DOI):

[10.2140/gt.2019.23.347](https://doi.org/10.2140/gt.2019.23.347)

Link:

[Link to publication record in Edinburgh Research Explorer](#)

Document Version:

Peer reviewed version

Published In:

Geometry & Topology

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Birational models of moduli spaces of coherent sheaves on the projective plane

Chunyi Li and Xiaolei Zhao

September 25, 2018

Abstract

In this paper, we study the birational geometry of moduli spaces of semistable sheaves on the projective plane via Bridgeland stability conditions. We show that the entire MMP of their moduli spaces can be run via wall-crossing. Via a description of the walls, we give a numerical description of their movable cones, along with its chamber decomposition corresponding to minimal models. As an application, we show that for primitive vectors, all birational models corresponding to open chambers in the movable cone are smooth and irreducible.

Introduction

Birational geometry of moduli space of sheaves on surfaces has been studied a lot in recent years, see [ABCH, BM2, BM3, BMW, CH1, CH2, CH3, CHW, LZ, Wo]. The milestone work in [BM2, BM3] completes the whole picture for K3 surfaces. In this paper, we give a complete description for the minimal model program of the moduli space of semistable sheaves on the projective plane via wall-crossings in the space of Bridgeland stability conditions. In particular, we deduce a description of their nef cone, movable cone and the chamber decomposition for their minimal models.

Geometric stability conditions on \mathbf{P}^2 : The notion of *stability condition* on a \mathbb{C} -linear triangulated category was first introduced in [Br1] by Bridgeland. A stability condition consists of a slicing \mathcal{P} of semistable objects in the triangulated category and a central charge Z on the Grothendieck group, which is compatible with the slicing. In particular in this paper, we consider the bounded derived category of coherent sheaves on the projective plane. A stability condition $\sigma = (Z, \mathcal{P})$ is called *geometric* if it satisfies the support property and all sky scraper sheaves are σ -stable with the same phase, see Definition 1.9.

The Grothendieck group $K(\mathbf{P}^2)$ of $D^b(\mathbf{P}^2)$ is of rank 3 and $K_{\mathbb{R}}(\mathbf{P}^2)(= K(\mathbf{P}^2) \otimes \mathbb{R})$ is spanned by the Chern characters ch_0 , ch_1 and ch_2 . Due to the work of Drezet and Le Potier, there is a *Le Potier cone* (see the picture below Definition 1.6) in the space $K_{\mathbb{R}}(\mathbf{P}^2)$, such that there exists slope stable coherent sheaves with character $w = (\text{ch}_0(> 0), \text{ch}_1, \text{ch}_2) \in K(\mathbf{P}^2)$ if and only if either w is the character of an exceptional bundle, or it is not inside the Le Potier cone. By taking the kernel of the central charge, the space of all geometric

stability conditions can be realized as a principal $\widetilde{\mathrm{GL}^+(2, \mathbb{R})}$ -bundle over Geo_{LP} , which is an open region above the *Le Potier curve*, see Proposition 1.13. Note that the $\widetilde{\mathrm{GL}^+(2, \mathbb{R})}$ -action does not affect the stability of objects. We will write a geometric stability condition as $\sigma_{s,q}$ with $(s, q) \in \mathrm{Geo}_{LP}$ indicating the kernel of its central charge. Let $\mathfrak{M}_{\sigma_{s,q}}^{s(ss)}(w)$ be the moduli space of $\sigma_{s,q}$ -(semi)stable objects in the heart $\mathrm{Coh}_{\#s}$ of the stability condition $\sigma_{s,q}$ with character $w \in K(\mathbf{P}^2)$, we address the following questions:

1. For a Chern character w and a geometric stability condition $\sigma_{s,q}$, when $\mathfrak{M}_{\sigma_{s,q}}^{ss}(w)$ is non-empty?
2. How does $\mathfrak{M}_{\sigma_{s,q}}^{ss}(w)$ change when $\sigma_{s,q}$ varies in Geo_{LP} ?

The first question is answered step by step in several parts of the paper. Similar to the result of Drezet and Le Potier, when the character w is inside the Le Potier cone and not exceptional (see Corollary 1.19), there is no $\sigma_{s,q}$ -semistable object with character w (or $-w$) for any geometric stability condition $\sigma_{s,q}$. In other words, $\mathfrak{M}_{\sigma_{s,q}}^{ss}(w)$ is always empty. When the character w is proportional to an exceptional character (see Corollary 1.30), both $\mathfrak{M}_{\sigma_{s,q}}^{ss}(w)$ and $\mathfrak{M}_{\sigma_{s,q}}^{ss}(-w)$ are empty if and only if: the point $(1, s, q)$ and the reduced character \tilde{w} are on different sides of the vertical line $L_{e\pm}$ for some exceptional character e , and $(1, s, q)$ is below the line L_{we} in the Geo_{LP} .

The main case of the first question is when the character is not inside the Le Potier cone.

Theorem 0.1 (Lemma 3.12, Theorem 3.14). *Let $w \in K(\mathbf{P}^2)$ be a character not inside the Le Potier cone, then $\mathfrak{M}_{\sigma_{s,q}}^{ss}(w)$ is empty if and only if $\sigma_{s,q}$ is not above L_w^{last} or $L_w^{\mathrm{right-last}}$ in the Geo_{LP} .*

The notations L_w^{last} and $L_w^{\mathrm{right-last}}$ are defined in Definition 3.11. As we will see, the description for the last wall is equivalent to that for the effective boundary of the moduli space. This is first solved in [CHW] for the $\mathrm{ch}_0 \geq 1$ case and in [Wo] for the torsion case by studying the effective cone of the moduli space. In this paper, we reprove these results in our set-up in a different way.

For the second question, we have the following result:

Theorem 0.2 (Theorem 2.19, Theorem 2.24). *Let w be a primitive character. The moduli space $\mathfrak{M}_{\sigma_{s,q}}^s(w)$ is smooth and connected for any generic geometric stability condition $\sigma_{s,q}$ when it is non-empty. Any two non-empty moduli spaces $\mathfrak{M}_{\sigma}^s(w)$ and $\mathfrak{M}_{\sigma'}^s(w)$ are birational to each other. The actual walls (chambers) is one-to-one corresponding to the stable base locus decomposition walls (chambers) of the divisor cone of $\mathfrak{M}_{GM}^s(w)$. In particular, one can run the whole minimal model program for $\mathfrak{M}_{GM}^s(w)$ via wall crossing in the space of geometric stability conditions.*

The smoothness result can be proved easily for moduli of slope stable sheaves. However, for Bridgeland stable objects, they may not remain stable after twisting by $\mathcal{O}(-3)$. The key point is to develop a method to compare slopes with respect to different Bridgeland stability conditions, and conclude the vanishing of Hom group. This is achieved first in [LZ], and generalized to the current situation in Section 2. The following consequence seems new to the theory of MMP of moduli of sheaves.

Corollary 0.3. *Let $w \in K(\mathbf{P}^2)$ be a primitive character not inside the Le Potier cone, then each minimal model of $\mathfrak{M}_{GM}^s(w)$ corresponding to an open chamber in the movable cone is smooth.*

The moduli space $\mathfrak{M}_{\sigma_{s,q}}^s(w)$ can be constructed via the geometric invariant theory, because there are the so-called *algebraic stability conditions*. The space Stab^{Alg} of algebraic stability conditions is large enough in the sense that for any Chern character w outside the Le Potier cone and stability condition $\sigma_{s,q}$ in Geo_{LP} , the line segment $l_{\sigma w}$ always intersects Stab^{Alg} . The moduli space $\mathfrak{M}_{\sigma'}^s(w)$ does not depend on the choice of σ' on $l_{\sigma w}$, we may assume that $\sigma_{s,q}$ is an algebraic stability condition. After a suitable homological shift, the heart of an algebraic stability condition is the same as the representation space of a path algebra with relations. The moduli space of such kind of quiver representations can be constructed via the geometric invariant theory. As an immediate corollary, the GIT construction ensures that $\mathfrak{M}_{\sigma_{s,q}}^{ss}(w)$ is projective. This construction first appears in [ABCH] for Hilbert schemes of point on \mathbf{P}^2 , and in [BMW] for moduli of sheaves, both using quivers associated to line bundles. In this paper, we further study the relationship between geometric stability and algebraic stability, and make available this construction for any full strong exceptional collections of \mathbf{P}^2 . This generalization will be important for some arguments in this paper.

When $q_0 \gg 0$ and $s_0 < \text{ch}_1(w)$, $\mathfrak{M}_{\sigma_{s_0,q_0}}^s(w)$ is the same as the moduli space $\mathfrak{M}_{GM}^s(w)$ of slope stable sheaves. For any Chern character v right orthogonal to w (i.e. $\chi(w, v) = 0$), the Donaldson morphism provides a divisor class \mathcal{L}_v in $\text{Pic}_{\mathbb{R}}(\mathfrak{M}_{GM}^s(w))$. On the other hand, for each moduli space $\mathfrak{M}_{\sigma_{s,q}}^s(w)$, the GIT construction provides a divisor class $[\mathcal{L}_{s,q}]$ in $\text{Pic}_{\mathbb{R}}(\mathfrak{M}_{\sigma_{s,q}}^s(w))$ up to a positive scalar. When the exceptional locus of the natural map $\mathfrak{M}_{\sigma_{s,q}}^s(w) \dashrightarrow \mathfrak{M}_{\sigma_{s_0,q_0}}^s(w) \simeq \mathfrak{M}_{GM}^s(w)$ (which is constructed via the variation of GIT concretely, see Section 2) has codimension greater than 1, the divisor class $[\mathcal{L}_{s,q}]$ is also defined on $\text{Pic}_{\mathbb{R}}(\mathfrak{M}_{GM}^s(w))$. Suppose the line $L_{w\sigma_{s,q}}$ is given by ${}^\perp v_{s,q}$ for some $v_{s,q} \in w^\perp$, then the divisor class $[\mathcal{L}_{s,q}]$ from the GIT construction is the same as $[\mathcal{L}_{v_{s,q}}]$ given by the Donaldson morphism up to a positive scalar.

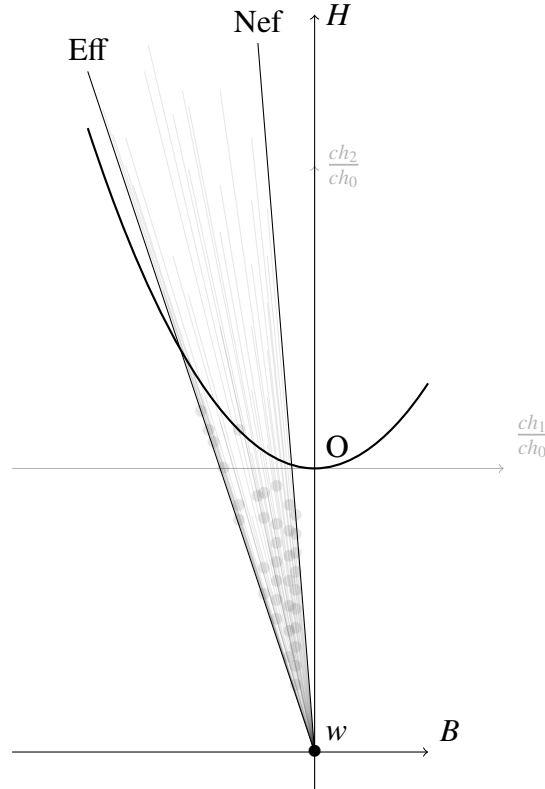
Based on the explicit correspondence between walls in the space of stability conditions and walls in the divisor cone, we may describe all stable base locus walls (including the boundaries of nef cone, effective cone and movable cone) as actual walls in the space of stability conditions. Here an actual wall for a Chern character w is a potential wall $L_{\sigma w}$ such that curves are contracted from either side of $\mathfrak{M}_{\sigma_\pm}^{ss}(w) \dashrightarrow \mathfrak{M}_{\sigma}^{ss}(w)$. So it becomes an important question to ask when a potential wall is an actual wall. In Section 3, we give a numerical criteria on actual walls, which depends on only numerical data, and provides an effective algorithm to compute all actual walls for w .

Theorem 0.4 (Theorem 3.16). *Let $w \in K(\mathbf{P}^2)$ be a Chern character with $\text{ch}_0(w) \geq 0$ not inside the Le Potier cone. For any stability condition σ inside $\bar{\Delta}_{<0} \subset \text{Geo}_{LP}$ between the last wall L_w^{last} and the vertical wall L_{w^\perp} , the wall $L_{\sigma w}$ is an actual wall for w if and only if there exists a Chern character $v \in K(\mathbf{P}^2)$ on the line segment $l_{\sigma w}$ such that: $\text{ch}_0(v) > 0$, $\frac{\text{ch}_1(w)}{\text{ch}_0(w)} > \frac{\text{ch}_1(v)}{\text{ch}_0(v)}$, the characters v and $w - v$ are either exceptional or not inside the Le Potier cone, and both of them are not in TR_{wE} for any exceptional bundle E .*

TR_{wE} is defined in Definition 3.15. It is a small triangle area decided by w and an exceptional character E . On an actual wall $L_{w\sigma}$, the Chern character w can always be written as the sum of two proper Chern characters w' and $w-w'$ satisfying the conditions in the theorem. The key is to prove the inverse direction: when such two characters exist, two stable objects with the corresponding characters extend to a stable object with character w , which is destabilized on the wall. Roughly speaking, three main steps are involved: 1. $\mathfrak{M}_\sigma^s(w')$ and $\mathfrak{M}_\sigma^s(w-w')$ are non-empty; 2. the extension group ext^1 between two generic objects in $\mathfrak{M}_\sigma^s(w')$ and $\mathfrak{M}_\sigma^s(w-w')$ is non-zero; 3. the extension of two stable objects will produce σ_+ or σ_- stable object with character w .

The conditions in the theorem are mainly used in step 1, and step 3 follows from general computations. For step 2, based on the characters, we can only aim to show $\chi(w-w', w') < 0$. However, one may wonder about the case that on an actual wall, generic objects in $\mathfrak{M}_\sigma^s(w')$ and $\mathfrak{M}_\sigma^s(w-w')$ do not have non-trivial extensions, but objects on some jumping loci extend to σ_\pm -stable objects. Should this happen, $\chi(w-w', w') \geq 0$ but objects in $\mathfrak{M}_\sigma^s(w')$ and $\mathfrak{M}_\sigma^s(w-w')$ still extend to σ_\pm -stable objects. From this point of view, it is a bit surprising to have a numerical criteria for actual walls. The main point in Theorem 0.4 is to rule out this possibility, and this is done by gaining a good understanding of the last wall.

Moreover, the criteria decides all the actual walls effectively, in the sense that it involves only finitely many steps of decisions, and one may write a computer program to output all the actual walls with a given Chern character w as the input. We compute the example for $w = (4, 0, -15)$ by hands to show some details of this computation. A cartoon for the actual walls in this case is as follows.



As two quick applications of Theorem 0.4, we decide the boundary (on the primitive side) of the nef cone and the movable cone of $\mathfrak{M}_{GM}^s(w)$ for a primitive character $w = (\text{ch}_0, \text{ch}_1, \text{ch}_2)$. The other side is dually decided by $w' = (\text{ch}_0, -\text{ch}_1, \text{ch}_2)$.

Theorem 0.5 (Theorem 4.3, the movable cone). *Let w be a primitive Chern character with $\text{ch}_0(w) \geq 0$ not inside the Le Potier cone. When $\chi(E, w) \neq 0$ for any exceptional bundle E with $\frac{\text{ch}_1(E)}{\text{ch}_0(E)} < \frac{\text{ch}_1(w)}{\text{ch}_0(w)}$, the movable cone boundary on the primitive side coincides with the effective cone boundary.*

When $\chi(E_\gamma, w) = 0$ for an exceptional bundle E_γ with $\frac{\text{ch}_1(E_\gamma)}{\text{ch}_0(E_\gamma)} < \frac{\text{ch}_1(w)}{\text{ch}_0(w)}$, let $E_\alpha, E_\beta, E_\gamma$ be exceptional bundles corresponding to dyadic numbers $\frac{p-1}{2^n}, \frac{p+1}{2^n}, \frac{p}{2^n}$ respectively, then w can be uniquely written as $n_2 e_\alpha - n_1 e_{\beta-3}$ for some positive integers n_1, n_2 . Define the character P accordingly as follows:

1. $P := e_\gamma - (3\text{ch}_0(E_\beta) - n_2)e_\alpha$, if $1 \leq n_2 < 3\text{ch}_0(E_\beta)$;
2. $P := e_\gamma$, if $n_2 \geq 3\text{ch}_0(E_\beta)$.

Then the wall L_{Pw} corresponds to the boundary of the movable cone of $\mathfrak{M}_{GM}^s(w)$.

Theorem 0.6 (Theorem 4.6, the nef cone). *Let w be a primitive Chern character with $\text{ch}_0(w) > 0$ and $\bar{\Delta}(w) \geq 10$, then the first actual wall to the left of vertical wall (i.e. nef cone boundary for $\mathfrak{M}_{GM}^s(w)$) is the first lower rank wall L_{vw} such that*

1. $\frac{\text{ch}_1(v)}{\text{ch}_0(v)}$ is the greatest rational number less than $\frac{\text{ch}_1(w)}{\text{ch}_0(w)}$ with $0 < \text{ch}_0(v) \leq \text{ch}_0(w)$;
2. given the first condition, if $\text{ch}_1(v)$ is even (odd resp.), let $\text{ch}_2(v)$ be the greatest integer ($2\text{ch}_2(v)$ be the greatest odd integer resp.), such that the point v is either an exceptional character or not inside Cone_{LP} .

The result on the nef cone is not hard to see from the $\{1, \frac{\text{ch}_1}{\text{ch}_0}, \frac{\text{ch}_2}{\text{ch}_0}\}$ -plane picture. First of all, when the Chern character w has certain distance from Geo_{LP} , the first wall is not of higher rank. On the $\{1, \frac{\text{ch}_1}{\text{ch}_0}, \frac{\text{ch}_2}{\text{ch}_0}\}$ -plane, although the Le Potier curve for the stable objects is zigzag, when the Chern character w is not very close to Geo_{LP} , the first wall L_{vw} is still given by a point v with $\frac{\text{ch}_1}{\text{ch}_0}$ -coordinate closest to the vertical wall.

The result on the movable cone is more subtle. When the Chern character w is right orthogonal to an exceptional bundle E_γ , the jumping locus

$$\{[F] \in \mathfrak{M}_{GM}^s(w) \mid \text{Hom}(E_\gamma, F) \neq 0\}$$

has codimension 1 and is the exceptional divisor that contracted on the movable cone boundary. However, the wall L_{we_γ} for $\text{Hom}(E_\gamma, F) \neq 0$ may not always be the wall for this contraction. In the case when $n_2 < 3\text{ch}_0(E_\beta)$, the exceptional divisor is already contracted at a wall prior to the wall L_{we_γ} . One simple example of such w is when $(\text{ch}_0, \text{ch}_1, \text{ch}_2) = (1, 0, -4)$, in other words, the ideal sheaf of four points. The Chern character w is right orthogonal to the cotangent bundle Ω , whose dyadic number is $-\frac{3}{2}$. The other exceptional bundles E_α and E_β are $\mathcal{O}(-2)$ and $\mathcal{O}(-1)$ respectively, and w can be written as $2[\mathcal{O}(-2)] - [\mathcal{O}(-4)]$. The jumping locus of $\text{Hom}(\Omega, w) \neq 0$ is the exceptional divisor, and

it is the same as the jumping locus $\text{Hom}(\mathcal{I}_1(-1), w) \neq 0$, where $\mathcal{I}_1(-1)$ stands for the ideal sheaf of one point tensor $\mathcal{O}(-1)$. Since the wall $L_{\mathcal{I}_1(-1)w}$ is between $L_{\Omega w}$ and L_{w^\pm} in the $\{1, \frac{\text{ch}_1}{\text{ch}_0}, \frac{\text{ch}_2}{\text{ch}_0}\}$ -plane, the boundary of movable cone should be given by $L_{\mathcal{I}_1(-1)w}$. Geometrically, the exceptional locus is where any three points are collinear.

Related Work. There are several papers [ABCH, BMW, CH1, CH2, CH3, CHW, LZ, Wo] studying the birational geometry of moduli of sheaves on the projective plane via wall crossing.

The study for Hilbert schemes of points on \mathbf{P}^2 first appears in [ABCH], and the wall crossing behavior is explicitly carried out for small numbers of points. It is also firstly suggested in [ABCH] that there is a correspondence between the wall crossing picture in the Bridgeland space and the minimal model program of the moduli space. In [CH1], the correspondence between walls in the Bridgeland space and stable locus decomposition walls in MMP is established for monomial schemes in the plane. In [LZ], we proved the full correspondence for Hilbert schemes of points, by establishing similar results as in Section 2 of this paper, and further generalize this correspondence to deformations of Hilbert schemes, or Hilbert schemes of non-commutative projective planes.

For moduli of torsion sheaves, the effective cone and the nef cone are computed in [Wo]. For general moduli of sheaves on \mathbf{P}^2 , the theory is built up in [BMW]. Among other results, the projectivity of moduli of Bridgeland stable objects is proved in [BMW]. The effective cone and the ample cone are computed in [CH2, CHW] respectively. Also [CHW] gives the criteria on when the movable cone coincides with the effective cone. We refer to the beautiful lecture notes [CH3] for details of these results.

Compared with these papers, the smoothness and irreducibility of moduli of Bridgeland stable objects with primitive characters are only proved in this paper, which enables us to deduce the equivalence between wall crossing and MMP for moduli of sheaves on \mathbf{P}^2 suggested in [ABCH]. Our result on the effective cone is essentially equivalent to that in [CHW], however, the proof is very different. Since this proof is very closely related to the proof of the criteria on actual walls, we decide to include it here. The numerical criteria on actual walls and the result on the movable cone are new. Our result on the nef cone (Theorem 0.6) follows from our numerical criteria. The nef cone was first proved in [CH2] when Δ is large enough with respect to ch_0 and $\frac{\text{ch}_1}{\text{ch}_0}$ (see Remark 8.7 in [CH2] for a lower bound), the bound in Theorem 0.6 is explicitly given by $\bar{\Delta} \geq 10$. Moreover, as a benefit of our set-up, in a large part of the paper we can treat the torsion case and the positive rank case uniformly. We make careful remarks on this through the paper.

Another important application of the wall-crossing machinery is towards the Le Potier strange duality conjecture. A special case is studied in [Ta].

Organization. In Section 1.1, we review some classical work by Drezet and Le Potier for stable sheaves on the projective plane. We prove some useful lemmas by visualizing the geometric stability conditions in the $\{1, \frac{\text{ch}_1}{\text{ch}_0}, \frac{\text{ch}_2}{\text{ch}_0}\}$ -plane in Section 1.3. These properties will play crucial roles for the arguments in the paper. In Section 2, we prove that the moduli space $\mathfrak{M}_\sigma^s(w)$ is smooth and irreducible for generic σ and primitive w . In this way, one can run the minimal model program for $\mathfrak{M}_{\text{GM}}^s(w)$ on the $\{1, \frac{\text{ch}_1}{\text{ch}_0}, \frac{\text{ch}_2}{\text{ch}_0}\}$ -plane. In Section 3, we first compute the last wall, and then prove the main theory in the paper: a criteria

for actual walls of $\mathfrak{M}_\sigma^s(w)$. In Section 4, we compute the nef and movable cone boundary as an application of the criteria for actual walls. Moreover, in Section 4.3, we work out a particular example for the character $(4, 0, -15)$.

Acknowledgments. The authors are greatly indebted to Arend Bayer, who offered tremendous assistance during the preparation of this work. We are grateful to Aaron Bertram, Izzet Coskun, Zheng Hua, Jack Huizenga, Wanmin Liu, Emanuele Macrì, Matthew Woolf and Ziyu Zhang for helpful conversations. We also had useful discussions with our advisors Herbert Clemens, Thomas Nevins and Karen Smith, and we would like to thank all of them. The author CL is supported by ERC starting grant no. 337039 “WallXBir-Geom”.

1 Stability conditions on $D^b(\mathbf{P}^2)$

In this section, we will recall some properties of the bounded derived category of coherent sheaves on the projective plane, and the construction of stability conditions on it. In Section 1.1, we will explain the structure of $D^b(\mathbf{P}^2)$ given by exceptional triples, and the numerical criteria on the existence of stable sheaves. A slice of the space of geometric stability conditions is discussed in Section 1.2, and the wall-chamber structure on it is studied in Section 1.3. In Section 1.4, we study the algebraic stability conditions, i.e. the stability conditions given by the exceptional triples. We also explain how they are glued to the slice of geometric stability conditions. In Section 1.5, we explain in detail the difference and advantage of our set-up over the one used in other papers. Finally in Section 1.6, we derive some easy numerical conditions on the existence of stable objects.

1.1 Review and notations: Exceptional objects, triples and the Le Potier curve

Let \mathcal{T} be a \mathbb{C} -linear triangulated category of finite type. In this article, \mathcal{T} will always be $D^b(\mathbf{P}^2)$: the bounded derived category of coherent sheaves on the projective plane over \mathbb{C} . We first recall the following definitions from [AKO, GR, Or].

Definition 1.1. *An object E in \mathcal{T} is called exceptional if*

$$\mathrm{Hom}(E, E[i]) = 0, \text{ for } i \neq 0; \mathrm{Hom}(E, E) = \mathbb{C}.$$

An ordered collection of exceptional objects $\mathcal{E} = \{E_0, \dots, E_m\}$ is called an exceptional collection if

$$\mathrm{Hom}(E_i, E_j[k]) = 0, \text{ for } i > j, \text{ any } k.$$

Definition 1.2. *Let $\mathcal{E} = \{E_0, \dots, E_n\}$ be an exceptional collection. We say this collection \mathcal{E} is strong, if*

$$\mathrm{Hom}(E_i, E_j[q]) = 0,$$

for all i, j and $q \neq 0$. This collection \mathcal{E} is called full, if \mathcal{E} generates \mathcal{T} under homological shifts, cones and direct sums.

1.1 Review and notations: Exceptional objects, triples and the Le Potier curve

An exceptional coherent sheaf on \mathbf{P}^2 is locally free since it is rigid. We summarize some results on the classification of exceptional bundles on \mathbf{P}^2 and introduce some notations, for details we refer to [DP, GR, LeP].

The Picard group of \mathbf{P}^2 is of rank one with generator $H = [O(1)]$, and we will, by abuse of notation, identify the i -th Chern character ch_i with its degree $H^{2-i}\text{ch}_i$. There is a one-to-one correspondence between exceptional bundles and dyadic integers, $\frac{p}{2^m}$, with integer p and non-negative integer m . Denote the exceptional bundle corresponding to $\frac{p}{2^m}$ by $E_{(\frac{p}{2^m})}$. We write Chern characters of $E_{(\frac{p}{2^m})}$ as

$$\tilde{v}\left(\frac{p}{2^m}\right) := \tilde{v}\left(E_{(\frac{p}{2^m})}\right) = \left(\text{ch}_0(E_{(\frac{p}{2^m})}), \text{ch}_1(E_{(\frac{p}{2^m})}), \text{ch}_2(E_{(\frac{p}{2^m})})\right).$$

They are inductively (on m) given by the formulas:

- $\tilde{v}(n) = (1, n, \frac{n^2}{2})$, for $n \in \mathbb{Z}$.
- When $m > 0$ and $p \equiv 3 \pmod{4}$, the Chern character is given by

$$\tilde{v}\left(\frac{p}{2^m}\right) = 3\text{ch}_0\left(E_{(\frac{p+1}{2^m})}\right)\tilde{v}\left(\frac{p-1}{2^m}\right) - \tilde{v}\left(\frac{p-3}{2^m}\right).$$

- When $m > 0$ and $p \equiv 1 \pmod{4}$, the character is given by

$$\tilde{v}\left(\frac{p}{2^m}\right) = 3\text{ch}_0\left(E_{(\frac{p-1}{2^m})}\right)\tilde{v}\left(\frac{p+1}{2^m}\right) - \tilde{v}\left(\frac{p+3}{2^m}\right).$$

Remark 1.3. *Here are some observations from the definition.*

1. $\tilde{v}(p)$ is the character of the line bundle $O(p) = E_{(p)}$.
2. $\tilde{v}(\frac{3}{2})$ is the character of the tangent sheaf $\mathcal{T}_{\mathbf{P}^2} = E_{(\frac{3}{2})}$.
3. The exceptional bundle $E_{(\frac{p}{2^m}+1)}$ is $E_{(\frac{p}{2^m})} \otimes O(1)$.
4. $\frac{\text{ch}_1(E_{(a)})}{\text{ch}_0(E_{(a)})} < \frac{\text{ch}_1(E_{(b)})}{\text{ch}_0(E_{(b)})}$ if and only if $a < b$.

For the rest of this section, we recall the construction of the Le Potier curve C_{LP} , which is greatly related to the existence of semistable sheaves.

The Grothendieck group $K(\mathbf{P}^2)$ has rank 3. We denote $K(\mathbf{P}^2) \otimes \mathbb{R}$ by $K_{\mathbb{R}}(\mathbf{P}^2)$. Consider the real projective space $P(K_{\mathbb{R}}(\mathbf{P}^2))$ with homogeneous coordinate $[\text{ch}_0, \text{ch}_1, \text{ch}_2]$, we view the locus $\text{ch}_0 = 0$ as the line at infinity. The complement forms an affine real plane, which is referred to as the $\{1, \frac{\text{ch}_1}{\text{ch}_0}, \frac{\text{ch}_2}{\text{ch}_0}\}$ -plane. We call $P(K_{\mathbb{R}}(\mathbf{P}^2))$ the projective $\{1, \frac{\text{ch}_1}{\text{ch}_0}, \frac{\text{ch}_2}{\text{ch}_0}\}$ -plane. For any object F in $D^b(\mathbf{P}^2)$, we write

$$\tilde{v}(F) := (\text{ch}_0(F), \text{ch}_1(F), \text{ch}_2(F))$$

as the (degrees of) Chern characters of F . When $\tilde{v}(F) \neq 0$, use $v(F)$ to denote the corresponding point in the projective $\{1, \frac{\text{ch}_1}{\text{ch}_0}, \frac{\text{ch}_2}{\text{ch}_0}\}$ -plane. In particular, when $\text{ch}_0(F) \neq 0$, $v(F)$ is in the $\{1, \frac{\text{ch}_1}{\text{ch}_0}, \frac{\text{ch}_2}{\text{ch}_0}\}$ -plane.

1.1 Review and notations: Exceptional objects, triples and the Le Potier curve

Remark 1.4. *In this article, in all arguments on the $\{1, \frac{\text{ch}_1}{\text{ch}_0}, \frac{\text{ch}_2}{\text{ch}_0}\}$ -plane, we assume the $\frac{\text{ch}_1}{\text{ch}_0}$ -axis to be horizontal and the $\frac{\text{ch}_2}{\text{ch}_0}$ -axis to be vertical. The term ‘above’ means ‘ $\frac{\text{ch}_2}{\text{ch}_0}$ coordinate is greater than’. Other terms such as ‘below’, ‘to the right’ and ‘to the left’ are understood in the similar way.*

Let $e(\frac{p}{2^m})$ be the point in the $\{1, \frac{\text{ch}_1}{\text{ch}_0}, \frac{\text{ch}_2}{\text{ch}_0}\}$ -plane with coordinate $(1, \frac{\text{ch}_1}{\text{ch}_0}(E(\frac{p}{2^m})), \frac{\text{ch}_2}{\text{ch}_0}(E(\frac{p}{2^m})))$. We associate to $E(\frac{p}{2^m})$ three points $e^+(\frac{p}{2^m})$, $e^l(\frac{p}{2^m})$ and $e^r(\frac{p}{2^m})$ in the $\{1, \frac{\text{ch}_1}{\text{ch}_0}, \frac{\text{ch}_2}{\text{ch}_0}\}$ -plane. The coordinate of $e^+(\frac{p}{2^m})$ is given by:

$$e^+\left(\frac{p}{2^m}\right) := v\left(E\left(\frac{p}{2^m}\right)\right) - \left(0, 0, \frac{1}{\left(\text{ch}_0\left(E\left(\frac{p}{2^m}\right)\right)\right)^2}\right).$$

For any real number a , let $\bar{\Delta}_a$ be the parabola:

$$\left\{ \left(1, \frac{\text{ch}_1}{\text{ch}_0}, \frac{\text{ch}_2}{\text{ch}_0}\right) \mid \bar{\Delta} := \frac{1}{2} \left(\frac{\text{ch}_1}{\text{ch}_0}\right)^2 - \frac{\text{ch}_2}{\text{ch}_0} = a \right\}$$

in the $\{1, \frac{\text{ch}_1}{\text{ch}_0}, \frac{\text{ch}_2}{\text{ch}_0}\}$ -plane. The point $e^l(\frac{p}{2^m})$ is defined to be the intersection of $\bar{\Delta}_{\frac{1}{2}}$ and the line segment $l_{e^+(\frac{p}{2^m})e^l(\frac{p-1}{2^m})}$, and $e^r(\frac{p}{2^m})$ is defined to be the intersection of $\bar{\Delta}_{\frac{1}{2}}$ and the line segment $l_{e^+(\frac{p}{2^m})e^r(\frac{p+1}{2^m})}$.

Remark 1.5. *In this paper we always use l_{**} to denote a line segment and L_{**} to denote a line in the $\{1, \frac{\text{ch}_1}{\text{ch}_0}, \frac{\text{ch}_2}{\text{ch}_0}\}$ -plane. Let E be the exceptional bundle $E(\frac{p}{2^m})$. The characters on the line $L_{e^+(\frac{p}{2^m})e^l(\frac{p}{2^m})e^l(\frac{p-1}{2^m})}$ satisfy the equation $\chi(E, -) = \chi(-, E(-3)) = 0$. Symmetrically, the line $L_{e^+(\frac{p}{2^m})e^r(\frac{p}{2^m})}$ is given by the equation $\chi(-, E) = \chi(E(3), -) = 0$ in the $\{1, \frac{\text{ch}_1}{\text{ch}_0}, \frac{\text{ch}_2}{\text{ch}_0}\}$ -plane.*

In the $\{1, \frac{\text{ch}_1}{\text{ch}_0}, \frac{\text{ch}_2}{\text{ch}_0}\}$ -plane, consider the open region below all the line segments $l_{e^+(\frac{p}{2^m})e^l(\frac{p}{2^m})}$, $l_{e^r(\frac{p}{2^m})e^+(\frac{p}{2^m})}$ and the curve $\bar{\Delta}_{\frac{1}{2}}$. The boundary of this open region is a fractal curve consisting of line segments $l_{e^+(\frac{p}{2^m})e^l(\frac{p}{2^m})}$, $l_{e^r(\frac{p}{2^m})e^+(\frac{p}{2^m})}$ for all dyadic numbers $\frac{p}{2^m}$ and fractal pieces of points on $\bar{\Delta}_{\frac{1}{2}}$. This curve is in the region between $\bar{\Delta}_{\frac{1}{2}}$ and $\bar{\Delta}_1$.

Definition 1.6. *The above boundary curve is called the Le Potier curve in the $\{1, \frac{\text{ch}_1}{\text{ch}_0}, \frac{\text{ch}_2}{\text{ch}_0}\}$ -plane, and denoted by C_{LP} . The cone in $\mathbf{K}_{\mathbb{R}}(\mathbf{P}^2)$ spanned by the origin and C_{LP} is defined to be the Le Potier cone, denoted by Cone_{LP} .*

We say a character $v \in \mathbf{K}(\mathbf{P}^2)$ is not inside Cone_{LP} if either $\text{ch}_0(v) \neq 0$ and the corresponding point \tilde{v} is not above C_{LP} in the $\{1, \frac{\text{ch}_1}{\text{ch}_0}, \frac{\text{ch}_2}{\text{ch}_0}\}$ -plane; or $\text{ch}_0(v) = 0$ and $\text{ch}_1 > 0$.

Remark 1.7. *The line segments $l_{e^+(\frac{p}{2^m})e^l(\frac{p}{2^m})}$, $l_{e^r(\frac{p}{2^m})e^+(\frac{p}{2^m})}$ do not cover the whole C_{LP} , the complement forms a Cantor set on $\bar{\Delta}_{\frac{1}{2}}$. The cartoon for C_{LP} in the $\{1, \frac{\text{ch}_1}{\text{ch}_0}, \frac{\text{ch}_2}{\text{ch}_0}\}$ -plane is shown as follows.*

1.2 Geometric stability conditions

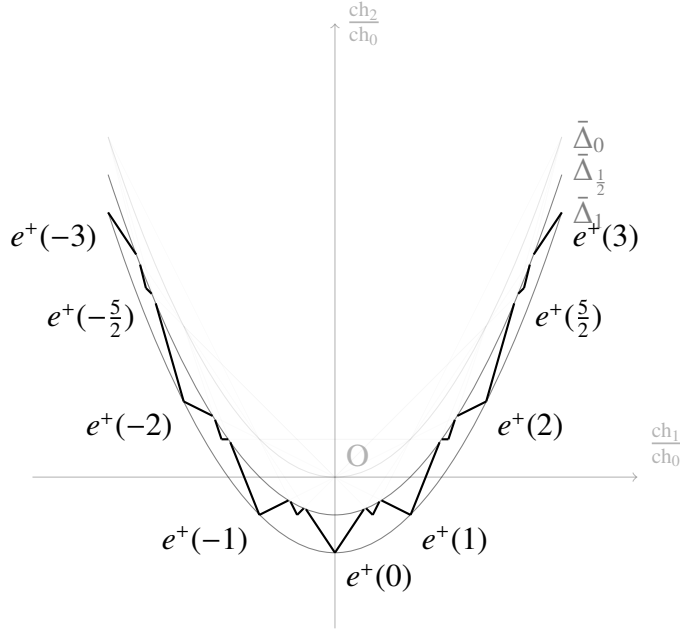


Figure: The Le Potier curve C_{LP} .

Given the Le Potier curve, we can now state the numerical condition on the existence of stable sheaves.

Theorem 1.8 (Drezet, Le Potier). *There exists a slope semistable coherent sheaf with character $w = (ch_0(> 0), ch_1, ch_2) \in K(\mathbf{P}^2)$ if and only if one of the following two conditions holds:*

1. w is proportional to an exceptional character;
2. The point $\left(1, \frac{ch_1}{ch_0}, \frac{ch_2}{ch_0}\right)$ is on or below C_{LP} in the $\{1, \frac{ch_1}{ch_0}, \frac{ch_2}{ch_0}\}$ -plane.

1.2 Geometric stability conditions

In this section, we follow [BM1, Br2] and recall that the space of geometric stability conditions on \mathbf{P}^2 is a $\widetilde{GL^+(2, \mathbb{R})}$ principal bundle over a subspace Geo_{LP} of the $\{1, \frac{ch_1}{ch_0}, \frac{ch_2}{ch_0}\}$ -plane.

In applications to geometry, the following type of stability conditions are always most relevant.

Definition 1.9. *A stability condition σ on $D^b(\mathbf{P}^2)$ is called geometric if it satisfies the support property and all skyscraper sheaves $k(x)$ are σ -stable of the same phase. We denote the set of all geometric stability conditions by $\text{Stab}^{\text{Geo}}(\mathbf{P}^2)$.*

In order to construct geometric stability conditions, we want to first introduce the appropriate t -structure. Fix a real number s , a torsion pair of coherent sheaves on \mathbf{P}^2 is given by:

$\text{Coh}_{\leq s}$: subcategory of $\text{Coh}(\mathbf{P}^2)$ generated by semistable sheaves of slope $\leq s$.

1.2 Geometric stability conditions

$\text{Coh}_{>s}$: subcategory of $\text{Coh}(\mathbf{P}^2)$ generated by semistable sheaves of slope $> s$ and torsion sheaves.

$$\text{Coh}_{\#s} := \langle \text{Coh}_{\leq s}[1], \text{Coh}_{>s} \rangle.$$

We define the *geometric area* Geo_{LP} in the $\{1, \frac{\text{ch}_1}{\text{ch}_0}, \frac{\text{ch}_2}{\text{ch}_0}\}$ -plane to be the open set:

$$\text{Geo}_{LP} := \{(1, a, b) \mid (1, a, b) \text{ is above } C_{LP} \text{ and not on } l_{ee^+} \text{ for any exceptional } e\}.$$

Proposition and Definition 1.10. *For a point $(1, s, q) \in \text{Geo}_{LP}$, there exists a geometric stability condition $\sigma_{s,q} := (Z_{s,q}, \text{Coh}_{\#s})$ on $\text{D}^b(\mathbf{P}^2)$, where the central charge is given by*

$$Z_{s,q}(E) := (-\text{ch}_2(E) + q \cdot \text{ch}_0(E)) + i(\text{ch}_1(E) - s \cdot \text{ch}_0(E)).$$

In this case, $\text{Ker}(Z_{s,q})$ consists of the characters corresponding to the point $(1, s, q)$. We write $\phi_{\sigma_{s,q}}$ or $\phi_{s,q}$ for the phase function of $\sigma_{s,q}$.

For the proof that $\sigma_{s,q}$ is indeed a geometric stability condition, we refer to [BM1] Corollary 4.6 and [Br2], which also work well for \mathbf{P}^2 . Here the phase function $\phi_{s,q}$ can be also defined for objects in $\text{Coh}_{\#s}$:

$$\phi_{s,q}(E) := \frac{1}{\pi} \text{Arg}(Z_{s,q}(E)).$$

It is well-defined in the sense that it coincides with the phase function on $\sigma_{s,q}$ -semistable objects.

Remark 1.11. *The definition of $\sigma_{s,q}$ here is different from the usual one as that in [ABCH], which is given as $(Z'_{s,t}, \mathcal{P}_s)$ (see Section 1.5 for the explicit formulae). When $q > \frac{s^2}{2}$, $Z_{s,q}$ has the same kernel as that of $Z'_{s,q-\frac{s^2}{2}}$. Their formula are slightly different. The imaginary parts are the same, but the real parts differ by a multiple of the imaginary part. We would like to use the version here because $q - \frac{s^2}{2}$ is allowed to be negative, and the kernel of the central charge on the $\{1, \frac{\text{ch}_1}{\text{ch}_0}, \frac{\text{ch}_2}{\text{ch}_0}\}$ -plane is clearly $(1, s, q)$.*

Remark 1.12. *Given a point $P = (1, s, q)$ in Geo_{LP} , we will also write σ_P , ϕ_P , $\text{Coh}_P(\mathbf{P}^2)$ and Z_P for the stability condition $\sigma_{s,q}$, the phase function $\phi_{s,q}$, the tilt heart $\text{Coh}_{\#s}(\mathbf{P}^2)$ and the central charge $Z_{s,q}$ respectively.*

Up to the $\widetilde{\text{GL}^+(2, \mathbb{R})}$ -action, geometric stability conditions are all of the form given in Proposition and Definition 1.10.

Proposition 1.13 ([Br2] Proposition 10.3, [BM1] Section 3). *Let $\sigma = (Z, \mathcal{P}((0, 1]))$ be a geometric stability condition such that all skyscraper sheaves $k(x)$ are contained in $\mathcal{P}(1)$. Then the heart $\mathcal{P}((0, 1])$ is $\text{Coh}_{\#s}$ for some real number s . The central charge Z can be written in the form of*

$$-\text{ch}_2 + a \cdot \text{ch}_1 + b \cdot \text{ch}_0,$$

where $a, b \in \mathbb{C}$ satisfy the following conditions:

- $\Im a > 0, \frac{\Im b}{\Im a} = s;$

1.3 Potential walls and phases

- $(1, \frac{\Im b}{\Im a}, \frac{\Re a \Im b}{\Im a} + \Re b)$ is in Geo_{LP} .

Thanks to the classification of characters of semistable sheaves on \mathbf{P}^2 [DP], this property is proved in the same way as in cases of local \mathbf{P}^2 [BM1] and K3 surfaces [Br1]. Since all discussions in this paper are invariant under the $\widetilde{\text{GL}^+(2, \mathbb{R})}$ -action, geometric stability conditions will be identified with the corresponding points in Geo_{LP} . We will always visualize $\text{Stab}^{\text{Geo}}(\mathbf{P}^2)$ as Geo_{LP} in this paper.

1.3 Potential walls and phases

We collect some well-known and useful results about the potential walls in this section. Since our set-up is slightly different from the usual one (see Remark 1.11), we give statements and proofs for completeness. We hope this can also illustrate the advantage of our set-up.

Definition 1.14. *A stability condition is said to be non-degenerate, if it satisfies the support property and the image of its central charge is not contained in any real line in \mathbb{C} . We write Stab^{nd} for the space of non-degenerate stability conditions.*

The kernel map for the central charges is well-defined on Stab^{nd} :

$$\text{Ker} : \text{Stab}^{\text{nd}} \rightarrow \text{P}_{\mathbb{R}}(\text{K}_{\mathbb{R}}(\mathbf{P}^2)).$$

Lemma 1.15 ([Br1]). *$\widetilde{\text{GL}^+(2, \mathbb{R})}$ acts freely on Stab^{nd} with closed orbits, and*

$$\text{Ker} : \text{Stab}^{\text{nd}} / \widetilde{\text{GL}^+(2, \mathbb{R})} \rightarrow \text{P}_{\mathbb{R}}(\text{K}_{\mathbb{R}}(\mathbf{P}^2))$$

is a local homeomorphism.

Proof. By [Br1], $\text{Stab}^{\text{nd}} \rightarrow \text{Hom}_{\mathbb{Z}}(\text{K}(\mathbf{P}^2), \mathbb{C})$ is a local homeomorphism, whose image lies in the subspace of non-degenerate morphisms in $\text{Hom}_{\mathbb{Z}}(\text{K}(\mathbf{P}^2), \mathbb{C})$. When taking the quotient by $\text{GL}^+(2, \mathbb{R})$, $\text{Hom}_{\mathbb{Z}}^{\text{nd}}(\text{K}(\mathbf{P}^2), \mathbb{C}) / \text{GL}^+(2, \mathbb{R})$ can be identified with the quotient Grassmannian $\text{Gr}_2(3) \cong \text{P}_{\mathbb{R}}(\text{K}_{\mathbb{R}}(\mathbf{P}^2))$ as a topological space. The statement clearly follows. \square

We have the following description of the potential wall, i.e. the locus of stability conditions for which two given characters are of the same slope.

Lemma 1.16 (Potential walls). *Let $P = (1, s, q)$ be a point in Geo_{LP} ; E and F be two objects in $\text{Coh}_P(\mathbf{P}^2)$ such that their Chern characters v and w are not zero, then*

$$Z_P(E) \text{ and } Z_P(F) \text{ are on the same ray}$$

if and only if v , w and P are collinear in the projective $\{1, \frac{\text{ch}_1}{\text{ch}_0}, \frac{\text{ch}_2}{\text{ch}_0}\}$ -plane.

Proof. $Z(v)$ and $Z(w)$ are on the same ray if and only if $Z(av - bw) = 0$ for some $a, b \in \mathbb{R}_+$. This happens only when v , w and $\text{Ker} Z$ are collinear in the projective $\{1, \frac{\text{ch}_1}{\text{ch}_0}, \frac{\text{ch}_2}{\text{ch}_0}\}$ -plane. \square

1.3 Potential walls and phases

Note that this statement holds even when v, w are torsion, i.e. $\text{ch}_0 = 0$.

We make some notations for lines and rays on the (projective) $\{1, \frac{\text{ch}_1}{\text{ch}_0}, \frac{\text{ch}_2}{\text{ch}_0}\}$ -plane. Consider objects E and F such that $v(E)$ and $v(F)$ are not zero, and let $\sigma_{s,q} = \sigma_P$ be a geometric stability condition. Let L_{EF} be the straight line on the projective $\{1, \frac{\text{ch}_1}{\text{ch}_0}, \frac{\text{ch}_2}{\text{ch}_0}\}$ -plane across $v(E)$ and $v(F)$. L_{EP} , as well as $L_{E\sigma}$, is the line across $v(E)$ and P . l_{EF} , as well as $l_{E\sigma}$, are the line segments on the $\{1, \frac{\text{ch}_1}{\text{ch}_0}, \frac{\text{ch}_2}{\text{ch}_0}\}$ -plane when both $v(E)$ and $v(F)$ are not at infinity. \mathcal{H}_P is the right half plane with either $\frac{\text{ch}_1}{\text{ch}_0} > s$, or $\frac{\text{ch}_1}{\text{ch}_0} = s$ and $\frac{\text{ch}_2}{\text{ch}_0} > q$. l_{PE}^+ is the ray along L_{PE} starting from P and completely contained in \mathcal{H}_P . $L_{E\pm}$ is the vertical wall $L_{E(0,0,1)}$. l_{E+} is the vertical ray along $L_{E(0,0,1)}$ from E going upward. l_{E-} is the vertical ray along $L_{E(0,0,-1)}$ from E going downward.

The following lemma translates the comparison of slopes into a geometric comparison of the positions of two rays. This simplifies a lot of computations and will be used throughout the paper.

Lemma 1.17. *Let $P = (1, s, q)$ be a point in Geo_{LP} , E and F be two objects in $\text{Coh}_{\#s}$. The inequality*

$$\phi_{s,q}(E) > \phi_{s,q}(F)$$

holds if and only if the ray l_{PE}^+ is above l_{PF}^+ .

Proof. By the formula of $Z_{s,q}$, the angle between the rays l_{PE}^+ and l_{PF}^+ at the point P is $\pi\phi_{s,q}(E)$. The statement follows from this observation. \square

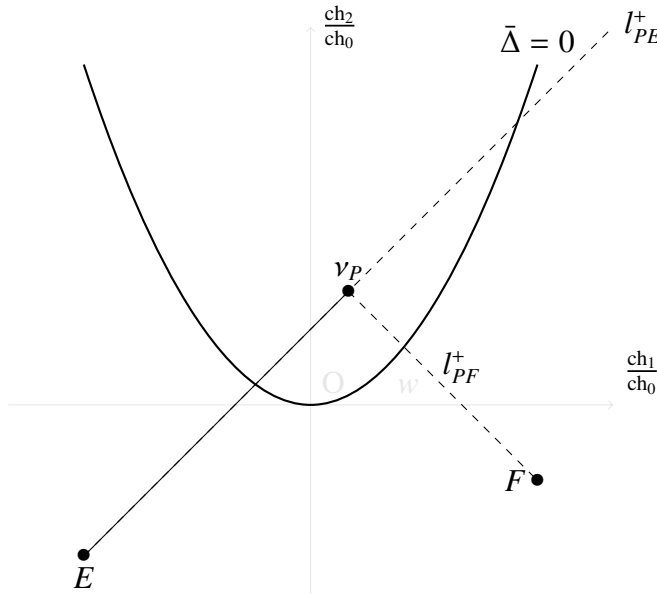


Figure: comparing the slopes at P .

An important problem is to study the existence of stable objects with respect to given stability condition and character. This will be solved in several steps in this paper. Now we can make the first observation.

Proposition 1.18. *Let $E \in \text{Coh}_{\#s}$ be a $\sigma_{s,q}$ -stable object, then one of the following cases happens:*

1.3 Potential walls and phases

1. The character $\tilde{v}(E)$ is not in the cone spanned by Geo_{LP} and the origin.
2. There exists a slope semistable sheaf F such that in the $\{1, \frac{\text{ch}_1}{\text{ch}_0}, \frac{\text{ch}_2}{\text{ch}_0}\}$ -plane the point $v(F)$ is above L_{EP} and between the vertical walls l_{E+} and l_{P+} .

In either case, the line segment l_{EP} is **not entirely contained inside** Geo_{LP} . In particular, at least one of $v(E)$ and $(1, s, q)$ is outside the negative discriminant area $\bar{\Delta}_{<0}$.

Proof. Assume that Case 1 does not happen, i.e. $\tilde{v}(E)$ is inside the Geo_{LP} -cone, we need to show case 2 happens. In particular, $\text{ch}_0(E)$ is not 0. When $\text{ch}_0(E) > 0$, $H^0(E)$ is non-zero. Let $F = H^0(E)_{\min}$ be the quotient sheaf of $H^0(E)$ with the minimum slope $\frac{\text{ch}_1}{\text{ch}_0}$. Then F is a slope semistable sheaf, so $v(F)$ is outside Geo_{LP} . Let D be $H^{-1}(E)$ and G be the kernel of $H^0(E) \rightarrow F$. We may compare the slopes of E and F

$$\frac{\text{ch}_1(E)}{\text{ch}_0(E)} = \frac{\text{ch}_1(F) + \text{ch}_1(G) - \text{ch}_1(D)}{\text{ch}_0(F) + \text{ch}_0(G) - \text{ch}_0(D)} \geq \frac{\text{ch}_1(F)}{\text{ch}_0(F)}.$$

The inequality holds because when D and G are non-zero, we have

$$\frac{\text{ch}_1(D)}{\text{ch}_0(D)} < \frac{\text{ch}_1(F)}{\text{ch}_0(F)} < \frac{\text{ch}_1(G)}{\text{ch}_0(G)}.$$

Note here the equality

$$\frac{\text{ch}_1(E)}{\text{ch}_0(E)} = \frac{\text{ch}_1(F)}{\text{ch}_0(F)}$$

holds only when D and G are both zero. In this case, $v(E) = v(F)$, hence $v(F)$ is inside Geo_{LP} , which contradicts to our assumption. Therefore, we have a strict inequality, i.e. $v(F)$ is to the left of $v(E)$. As $F \in \text{Coh}_{>s}$, P is to the left of $v(F)$. In addition, as $\phi_{s,q}(E) < \phi_{s,q}(F)$, by Lemma 1.17, F is above l_{PE} , so case 2 happens.

When $\text{ch}_0(E) < 0$, let $F = H^{-1}(E)_{\max}$ be the subsheaf of $H^{-1}(E)$ with the maximum slope $\frac{\text{ch}_1}{\text{ch}_0}$. By the same argument, $v(F)$ is to the right of $v(E)$. As $F \in \text{Coh}_{\leq s}$, v is to the left of $L_{P\pm}$ or on the ray l_{P-} . In addition, since $\phi_{s,q}(F[1]) < \phi_{s,q}(E)$, by Lemma 1.17, F is above l_{EP} . As l_{F-} does not intersect Geo_{LP} , F is to the left of $L_{P\pm}$.

For the last statement, since the region $\bar{\Delta}_{<0}$ is convex, for any $v(E)$ and $P = (1, s, q)$ that are both in $\bar{\Delta}_{<0}$, the line segment l_{EP} is also in $\bar{\Delta}_{<0}$, which is contained in Geo_{LP} . \square

This induces some useful corollaries. First we get the stability of exceptional bundles for some stability conditions.

Corollary 1.19. *Let E be an exceptional bundle, and $P = (1, s, q)$ be a point in Geo_{LP} , then E is $\sigma_{s,q}$ -stable if $s < \frac{\text{ch}_1}{\text{ch}_0}(E)$ and l_{EP} is contained in Geo_{LP} (not include the end-points). In the homological shifted case, $E[1]$ is $\sigma_{s,q}$ -stable, if $s \geq \frac{\text{ch}_1}{\text{ch}_0}(E)$ and l_{EP} is contained in Geo_{LP} .*

Proof. We will prove the first statement. If E is not $\sigma_{s,q}$ -stable, then there is a $\sigma_{s,q}$ -stable object F destabilizing E . We have the exact sequence

$$0 \rightarrow H^{-1}(F) \rightarrow H^{-1}(E) \rightarrow H^{-1}(E/F) \rightarrow H^0(F) \rightarrow H^0(E) \rightarrow H^0(E/F) \rightarrow 0.$$

1.4 Algebraic stability conditions

Since $H^{-1}(E) = 0$, we see that $H^{-1}(F) = 0$ and $v(F)$ lies between the vertical lines $L_{P\pm}$ and $L_{E\pm}$. Since $\phi_{s,q}(F) > \phi_{s,q}(E)$, by Lemma 1.17, $v(F)$ is in the region bounded by l_{P+} , l_{PE} and l_{E+} . As l_{EP} is contained in Geo_{LP} , l_{FP} is also contained in Geo_{LP} . By Proposition 1.18, F is not $\sigma_{s,q}$ -stable, which is a contradiction. The second statement can be proved similarly. \square

Remark 1.20. *The condition that ' l_{EP} is contained in Geo_{LP} ' is also necessary. Any ray starting from $v(E)$ may only intersect C_{LP} at most once, and only intersect with finitely many l_{ee+} segments. Suppose that $s < \frac{\text{ch}_1}{\text{ch}_0}(E)$, and l_{EP} intersects some l_{ee+} segments, we may choose the one (denoted by F) with minimum $\frac{\text{ch}_1}{\text{ch}_0}$ -coordinate. The segment l_{FP} is contained in Geo_{LP} , and $\phi_{s,q}(F) > \phi_{s,q}(E)$. By Corollary 1.19, F is $\sigma_{s,q}$ -stable. By [GR], $\text{Hom}(F, E) \neq 0$ when $\frac{\text{ch}_1}{\text{ch}_0}(F) < \frac{\text{ch}_1}{\text{ch}_0}(E)$. This shows that E is not $\sigma_{s,q}$ -stable.*

The second corollary roughly says when we vary the stability condition in Geo_{LP} , stable objects remain stable if the slopes do not change.

Corollary 1.21. *Let $\sigma_{s,q}$ be a geometric stability condition and F be a σ -stable object, then for any geometric stability condition τ on the line $L_{F\sigma}$ such that $l_{\tau\sigma}$ is contained in Geo_{LP} , F is also τ -stable.*

1.4 Algebraic stability conditions

The structure of $D^b(\mathbf{P}^2)$ can be studied via full strong exceptional collections. First recall the following definition.

Definition 1.22. *An ordered set $\mathcal{E} = \{E_1, E_2, E_3\}$ is an exceptional triple in $D^b(\mathbf{P}^2)$ if \mathcal{E} is a full strong exceptional collection of coherent sheaves in $D^b(\mathbf{P}^2)$.*

Remark 1.23. *The exceptional triples in $D^b(\mathbf{P}^2)$ have been classified in [GR] by Goro-dentsev and Rudakov. In particular, up to a cohomological shift, each collection consists of exceptional bundles on \mathbf{P}^2 . In terms of dyadic numbers, their labels are given by one of the following three cases (p is an odd integer when $m \neq 0$):*

$$\left\{ \frac{p-1}{2^m}, \frac{p}{2^m}, \frac{p+1}{2^m} \right\}; \left\{ \frac{p}{2^m}, \frac{p+1}{2^m}, \frac{p-1}{2^m} + 3 \right\}; \left\{ \frac{p+1}{2^m} - 3, \frac{p-1}{2^m}, \frac{p}{2^m} \right\}. \quad (\clubsuit)$$

We recall the construction of algebraic stability conditions associated to an exceptional triple.

Proposition 1.24 ([Ma] Section 3). *Let \mathcal{E} be an exceptional triple in $D^b(\mathbf{P}^2)$. For any positive real numbers m_1, m_2, m_3 and real numbers ϕ_1, ϕ_2, ϕ_3 such that:*

$$\phi_1 < \phi_2 < \phi_3, \text{ and } \phi_1 + 1 < \phi_3,$$

there exists a unique stability condition $\sigma = (Z, \mathcal{P})$ such that

1. E_j 's are σ -stable of phase ϕ_j ;

1.4 Algebraic stability conditions

$$2. Z(E_j) = m_j e^{\pi i \phi_j}.$$

Definition 1.25. Let \mathcal{E} be an exceptional triple $\{E_1, E_2, E_3\}$ in $D^b(\mathbf{P}^2)$, we write $\mathcal{A}_{\mathcal{E}}$ for the heart $\langle E_1[2], E_2[1], E_3 \rangle$, and $\Theta_{\mathcal{E}}$ for the space of all stability conditions in Proposition 1.24. $\Theta_{\mathcal{E}}$ is parametrized by

$$\{(m_1, m_2, m_3, \phi_1, \phi_2, \phi_3) \in (\mathbb{R}_{>0})^3 \times \mathbb{R}^3 \mid \phi_1 < \phi_2 < \phi_3, \phi_1 + 1 < \phi_3\}.$$

We consider the following two subsets of $\Theta_{\mathcal{E}}$.

- $\Theta_{\mathcal{E}}^{\nabla} := \{\sigma \in \Theta_{\mathcal{E}} \mid \phi_2 - \phi_1 < 1, \phi_3 - \phi_2 < 1\};$
- $\Theta_{\mathcal{E}}^{\text{Geo}} := \Theta_{\mathcal{E}} \cap \text{Stab}^{\text{Geo}};$

We denote Stab^{Alg} as the union of $\Theta_{\mathcal{E}}$ for all exceptional triples in $D^b(\mathbf{P}^2)$. A stability condition in Stab^{Alg} is called an algebraic stability condition.

Let $\text{TR}_{\mathcal{E}}$ be the inner points in the triangle bounded by $l_{e_1 e_2}$, $l_{e_2 e_3}$ and $l_{e_3 e_1}$ in the $\{1, \frac{\text{ch}_1}{\text{ch}_0}, \frac{\text{ch}_2}{\text{ch}_0}\}$ -plane, for $1 \leq i < j \leq 3$. Let e_i^* be the points associated to e_i defined in the first section, where $i = 1, 2, 3$ and $*$ could be $+$, l , or r . The points e_1^+, e_1^r, e_2, e_3 are on the line $\chi(-, E_1) = 0$, and e_3^+, e_3^l, e_2, e_1 are on the line $\chi(E_3, -) = 0$. Let $\text{MZ}_{\mathcal{E}}$ be the inner points of the region bounded by the line segments $l_{e_1 e_1^+}, l_{e_1^+ e_2}, l_{e_2 e_3^+}, l_{e_3^+ e_3}$ and $l_{e_3 e_1}$.

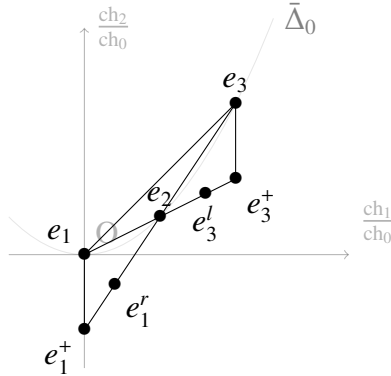


Figure: $\text{TR}_{\mathcal{E}}$ and $\text{MZ}_{\mathcal{E}}$.

The next proposition explains how the algebraic part $\Theta_{\mathcal{E}}$ ‘glues’ onto the geometric part Stab^{Geo} .

Proposition 1.26. Let \mathcal{E} be an exceptional triple, then we have:

1. $\Theta_{\mathcal{E}}^{\nabla} = \text{GL}^+(2, \mathbb{R}) \cdot \{\sigma_{s,q} \in \text{Stab}^{\text{Geo}}(\mathbf{P}^2) \mid (1, s, q) \in \text{TR}_{\mathcal{E}}\}.$
2. $\Theta_{\mathcal{E}}^{\text{Geo}} = \text{GL}^+(2, \mathbb{R}) \cdot \{\sigma_{s,q} \in \text{Stab}^{\text{Geo}}(\mathbf{P}^2) \mid (1, s, q) \in \text{MZ}_{\mathcal{E}}\}.$

In particular, $\Theta_{\mathcal{E}}^{\nabla}$ is contained in $\Theta_{\mathcal{E}}^{\text{Geo}}$.

1.5 Remarks on the $\{1, \frac{\text{ch}_1}{\text{ch}_0}, \frac{\text{ch}_2}{\text{ch}_0}\}$ -plane

Proof. We will first prove the second statement. As $\text{MZ}_{\mathcal{E}}$ is contained in Geo_{LP} , by Corollary 1.19, E_2 or $E_2[1]$ is $\sigma_{s,q}$ -stable for any point $(1, s, q)$ in $\text{MZ}_{\mathcal{E}}$. As e_1^+, e_1^r, e_2, e_3 are collinear on the line of $\chi(-, E_1) = 0$, for any point P in $\text{MZ}_{\mathcal{E}}$, l_{E_3P} is contained in Geo_{LP} . By Corollary 1.19, E_3 is stable for any stability conditions in $\text{MZ}_{\mathcal{E}}$. For the same reason, $E_1[1]$ is stable for any stability conditions in $\text{MZ}_{\mathcal{E}}$.

For any $(1, s, q)$ in $\text{MZ}_{\mathcal{E}}$, E_3 and $E_1[1]$ are in the heart $\text{Coh}_{\#s}$. By Lemma 1.17, $\phi_{s,q}(E_1[1]) < \phi_{s,q}(E_3)$, hence

$$\phi_{s,q}(E_3) - \phi_{s,q}(E_1) > 1.$$

When $s \geq \frac{\text{ch}_1}{\text{ch}_0}(E_2)$, E_3 and $E_2[1]$ are in the heart $\text{Coh}_{\#s}$, we have

$$\phi_{s,q}(E_3) - \phi_{s,q}(E_2) > 0.$$

As $(1, s, q)$ is above $L_{e_1 e_2}$, by Lemma 1.17, we also have

$$\phi_{s,q}(E_2) - \phi_{s,q}(E_1) > 0.$$

When $s < \frac{\text{ch}_1}{\text{ch}_0}(E_2)$, by a similar argument we have the same inequalities for $\phi_{s,q}(E_i)$'s. By Proposition 1.24, we get the embedding

$$\text{Ker}^{-1}(\text{MZ}_{\mathcal{E}}) \cap \text{Stab}^{\text{Geo}} \hookrightarrow \Theta_{\mathcal{E}} \cap \text{Stab}^{nd} \xrightarrow{\text{Ker}} \text{P}(\text{K}_{\mathbb{R}}(\mathbf{P}^2)).$$

For $(1, s, q)$ outside the area $\text{MZ}_{\mathcal{E}}$, by Lemma 1.17, at least one of the inequalities:

$$\phi_{s,q}(E_2) \leq \phi_{s,q}(E_1), \quad \phi_{s,q}(E_3) \leq \phi_{s,q}(E_2), \quad \text{or} \quad \phi_{s,q}(E_3) - \phi_{s,q}(E_1) \leq 1$$

holds. Hence $\sigma_{s,q}$ is not contained in $\Theta_{\mathcal{E}}$, the second statement of the proposition holds.

For statement 1, as $\phi_2 - \phi_1$ is not an integer, $\Theta_{\mathcal{E}}^{\vee} \in \text{Stab}^{nd}$. The image of $\text{Ker}(\Theta_{\mathcal{E}}^{\vee})$ is in $\text{TR}_{\mathcal{E}}$. By the previous argument, we also have the embedding

$$(\text{Ker}^{-1}(\text{TR}_{\mathcal{E}}) \cap \text{Stab}^{\text{Geo}}) / \widetilde{\text{GL}^+(2, \mathbb{R})} \hookrightarrow \Theta_{\mathcal{E}}^{\vee} / \widetilde{\text{GL}^+(2, \mathbb{R})} \xrightarrow{\text{Ker}} \text{TR}_{\mathcal{E}} \subset \text{P}(\text{K}_{\mathbb{R}}(\mathbf{P}^2)).$$

The map Ker is a local homeomorphism and the composition is an isomorphism. Since $\Theta_{\mathcal{E}}^{\vee}$ is path connected, the two maps are both isomorphisms. We get the first statement of the proposition. \square

1.5 Remarks on the $\{1, \frac{\text{ch}_1}{\text{ch}_0}, \frac{\text{ch}_2}{\text{ch}_0}\}$ -plane

In this section, we want to summarize some properties of our $\{1, \frac{\text{ch}_1}{\text{ch}_0}, \frac{\text{ch}_2}{\text{ch}_0}\}$ -plane from previous sections. The aim is to help the readers gain a better understanding, especially those who are already familiar with the classical (s, t) -upper half plane model.

The set-up of the space of stability conditions in the paper is different from the classical (s, t) -upper half plane model. Recall that we visualize a geometric stability condition as the kernel of its central charge in $\text{K}(\mathbf{P}^2) \otimes \mathbb{R}$. In particular, when the central charge is non-degenerate, which is always the case for geometric stability conditions, the kernel is a straight line in $\text{K}(\mathbf{P}^2) \otimes \mathbb{R}$. We further take the projectivization of $\text{K}(\mathbf{P}^2) \otimes \mathbb{R}$, the kernel of the central charge is a point on $\text{P}(\text{K}_{\mathbb{R}}(\mathbf{P}^2))$. For a geometric stability condition, to satisfies

1.5 Remarks on the $\{1, \frac{\text{ch}_1}{\text{ch}_0}, \frac{\text{ch}_2}{\text{ch}_0}\}$ -plane

the Harder-Narasimhan condition, the kernel of the central charge has to separate away from all the slope stable characters and torsion characters. In particular, the kernel can only be in the area Geo_{LP} bounded by the Le Potier curve. The $\widetilde{\text{GL}^+(2, \mathbb{R})}$ action does not affect the kernel of the central charge, and the space of the geometric stability condition is realized as a $\widetilde{\text{GL}^+(2, \mathbb{R})}$ -principal bundle over Geo_{LP} .

For a point in Geo_{LP} with coordinate $(1, s, q)$, we may write down a stability condition $\sigma_{s,q} = (Z_{s,q}, \mathcal{P}_s)$ with heart $\mathcal{P}_s((0, 1]) = \text{Coh}_{\#s}$ and central charge as that in Proposition and Definition 1.10:

$$Z_{s,q} = -(\text{ch}_2 - q \cdot \text{ch}_0) + i(\text{ch}_1 - s \cdot \text{ch}_0).$$

In many of other papers, a family of geometric stability condition is parameterized by (s, t) on the upper half plane \mathbb{H} via $\sigma'_{s,t} = (Z'_{s,t}, \mathcal{P}_s)$ with the same heart $\mathcal{P}_s((0, 1]) = \text{Coh}_{\#s}$ and a different central charge

$$Z'_{s,t} = -(\text{ch}_2^s + \frac{t^2}{2} \cdot \text{ch}_0) + i t \text{ch}_1^s.$$

Up to the $\widetilde{\text{GL}^+(2, \mathbb{R})}$ action, $\sigma'_{s,t}$ is the same as $\sigma_{s, \frac{s^2+t^2}{2}}$. Note that under this correspondence, the (s, t) -upper half plane \mathbb{H} is mapped to $\left\{ (1, \frac{\text{ch}_1}{\text{ch}_0}, \frac{\text{ch}_2}{\text{ch}_0}) \mid \left(\frac{\text{ch}_1}{\text{ch}_0}\right)^2 - 2\left(\frac{\text{ch}_2}{\text{ch}_0}\right) < 0 \right\}$ in the $\{1, \frac{\text{ch}_1}{\text{ch}_0}, \frac{\text{ch}_2}{\text{ch}_0}\}$ -plane in $\text{P}(\text{K}_{\mathbb{R}}(\mathbf{P}^2))$.

Since this different convention may upset some readers, we want to briefly illustrate some advantages of our approach, which will become more clear later in the paper. One most important benefit is that the characters and the stability conditions are on a same space. As seen in Section 1.3, the potential wall of w and another Chern character v is the straight line across these two points on the $\{1, \frac{\text{ch}_1}{\text{ch}_0}, \frac{\text{ch}_2}{\text{ch}_0}\}$ -plane, or strictly speaking, the line segment in Geo_{LP} . On the usual (s, t) upper half, the potential wall is the semicircle with two endpoints being $L_{vw} \cap \bar{\Delta}_0$. Let σ_P be a stability condition and w be a Chern character on the $\{1, \frac{\text{ch}_1}{\text{ch}_0}, \frac{\text{ch}_2}{\text{ch}_0}\}$ -plane, the argument of $Z_P(w)$ is the angle bounded by L_{P-} and L_{Pw} . We may compare the slopes of different Chern characters by their positions on the $\{1, \frac{\text{ch}_1}{\text{ch}_0}, \frac{\text{ch}_2}{\text{ch}_0}\}$ -plane and this reduces huge amount of computations. This allows us to deal with several Chern characters and stability conditions simultaneously.

Moreover, in our set-up, the divisor cone can be identified with the $\{1, \frac{\text{ch}_1}{\text{ch}_0}, \frac{\text{ch}_2}{\text{ch}_0}\}$ -plane. For a Chern character w , one may draw its $\text{Pic}_{\mathbb{R}}(\mathfrak{M}_{\sigma}^s(w))$ as a HB -coordinate (H vertical-axis; B with slope $\frac{\text{ch}_1}{\text{ch}_0}$) with origin at w on the $\{1, \frac{\text{ch}_1}{\text{ch}_0}, \frac{\text{ch}_2}{\text{ch}_0}\}$ -plane, the actual walls are the base locus decomposition walls. The Donaldson morphism identifies w^{\perp} with the divisor cone of $\mathfrak{M}_{\text{GM}}^s(w)$. Let v belong to w^{\perp} , then the divisor given by v via the Donaldson morphism corresponds to the wall $\chi(-, v) = 0$ on the $\{1, \frac{\text{ch}_1}{\text{ch}_0}, \frac{\text{ch}_2}{\text{ch}_0}\}$ -plane. The cartoon for the Chern character $(4, 0, -15)$ in the introduction can now be interpreted from this new viewpoint.

Another advantage of the $\{1, \frac{\text{ch}_1}{\text{ch}_0}, \frac{\text{ch}_2}{\text{ch}_0}\}$ -plane picture is that the space Geo_{LP} is larger than the usual upper half plane. As explained previously, up to the $\widetilde{\text{GL}^+(2, \mathbb{R})}$ action, Geo_{LP} is the whole space of geometric stability conditions. The algebraic stability conditions (quiver regions) for exceptional triples are also easier to understand on the $\{1, \frac{\text{ch}_1}{\text{ch}_0}, \frac{\text{ch}_2}{\text{ch}_0}\}$ -plane rather than on the upper half plane. The quiver region with heart $\langle E_1[2], E_2[1], E_3 \rangle$ in

1.6 First constraint on the last wall

the $\{1, \frac{\text{ch}_1}{\text{ch}_0}, \frac{\text{ch}_2}{\text{ch}_0}\}$ -plane is the area that is below $l_{e_1 e_3}$ and above $l_{e_2 e_1^+}, l_{e_2 e_3^+}$. Since Chern characters of exceptional bundles are usually not on the parabola $\bar{\Delta}_0$ (this is the case only for line bundles), the end points of the semicircular potential walls of them involve complicated computation. On the (s, t) -upper half plane, only quiver regions for heart $\langle O(k-1)[2], O(k)[1], O(k+1) \rangle$ can be neatly described. In this paper, we need the general quiver regions (e.g. for heart $\langle O(1)[2], \mathcal{T}[1], O(2) \rangle$), which are important to decide the stable area for exceptional characters, and are useful to understand the effective and movable cone boundary of the $\mathcal{M}_\sigma^s(w)$. So the $\{1, \frac{\text{ch}_1}{\text{ch}_0}, \frac{\text{ch}_2}{\text{ch}_0}\}$ -plane seems to be a suitable choice.

1.6 First constraint on the last wall

For a character, it is important to study the set of stability conditions for which there exist stable objects of the given character. We call this set the *stable area* of the character. In this section, we give a first constraint on the stable area.

Proposition 1.27. *Let w be a Chern character such that $\text{ch}_0(w) > 0$ and $\bar{\Delta}(w) > 0$, and E be an exceptional bundle such that $\frac{\text{ch}_1(E)}{\text{ch}_0(E)} < \frac{\text{ch}_1(w)}{\text{ch}_0(w)}$. Suppose w is above the line $L_{e^l e^+}$ in the $\{1, \frac{\text{ch}_1}{\text{ch}_0}, \frac{\text{ch}_2}{\text{ch}_0}\}$ -plane, then for any point $P \in \text{Geo}_{LP}$ below L_{wE} and to the left of $L_{E\pm}$, there is no σ_P -semistable object F with Chern character w .*

Remark 1.28. *When $\text{ch}_0(w) = 0$, there is a similar statement. The conditions are replaced by ' $\text{ch}_1(w) > 0$ ' and ' $\frac{\text{ch}_2}{\text{ch}_1}(w)$ is greater than the slope of $L_{e^l e^+}$ '. The proof is similar and left to the readers.*

Proof. By the assumptions and Corollary 1.21, we may assume that P is in $\text{MZ}_\mathcal{E}$ for an exceptional triple $\mathcal{E} = \{E_1, E_2, E_3\}$ such that $\frac{\text{ch}_1}{\text{ch}_0}(E_3) \leq \frac{\text{ch}_1}{\text{ch}_0}(E)$ and e_3 is above l_{Pw} . By an easy geometric property of C_{LP} , the character w is also above $L_{e_3^l e_3^+}$. E_3 satisfies the assumptions, without loss of generality, we may assume that $E_3 = E$.

We argue by contradiction. Assume F is a σ_P -stable object with Chern character w . As σ_P is below L_{wE} , by Lemma 1.17, we have

$$\phi_P(F) < \phi_P(E).$$

Since E and F are both σ_P -semistable, we have

$$\text{Hom}(E, F) = 0.$$

On the other hand, since P is in $\text{MZ}_\mathcal{E}$, it is to the right of $L_{E(-3)\pm}$. Therefore, $E(-3)[1]$ and F are in a same heart, we have

$$(\text{Hom}(E, F[2]))^* = \text{Hom}(F, E(-3)) = \text{Hom}(F, E(-3)[1][-1]) = 0.$$

The two Hom vanishings imply $\chi(E, F) \leq 0$. But by assumptions that $\text{ch}_0(F) \geq 0$, and w is above the line $L_{e^l e^+}$, which is given by $\chi(E, -) = 0$, we have $\chi(E, F) > 0$. This leads to a contradiction. \square

Remark 1.29. *The symmetric statement for w with $\text{ch}_0(w) < 0$ above $L_{e^+ e^r}$ and for E with larger $\frac{\text{ch}_1}{\text{ch}_0}$ can be proved in the same way.*

Now we have the following result on characters of Bridgeland stable objects. Note that this is a generalization of Theorem 1.8 to Bridgeland stable objects.

Corollary 1.30. *Fix a character w . Suppose that there exist $\sigma_{s,q}$ -semistable objects of character w for some geometric stability condition $\sigma_{s,q}$. Then w either lies not inside Cone_{LP} or is proportional to an exceptional character.*

Proof. Suppose w is inside Cone_{LP} and not proportional to any exceptional character, by Proposition 1.18, there is an exceptional character e such that e is above the line segment $l_{w\sigma}$ and between vertical walls $L_{w\pm}$ and $L_{\sigma\pm}$. We may assume that $\frac{\text{ch}_1(w)}{\text{ch}_0(w)} > s$, then w is above L_{e^+} . Now by Proposition 1.27, since σ is below L_{we} and to the left of $L_{e\pm}$, there is no σ -semistable object of character w , which is a contradiction. \square

We also want to introduce the following important notion.

Definition 1.31. *Let L be a straight line in the $\{1, \frac{\text{ch}_1}{\text{ch}_0}, \frac{\text{ch}_2}{\text{ch}_0}\}$ -plane. Suppose L intersects with $\bar{\Delta}_{\leq 0}$ along a line segment with two endpoints: $(1, f_1, g_1)$ and $(1, f_2, g_2)$. The $\frac{\text{ch}_1}{\text{ch}_0}$ -length of $L \cap \bar{\Delta}_{\leq 0}$ is defined to be $|f_1 - f_2|$.*

In the (s, t) -upper half plane model in [ABCH], the $\frac{\text{ch}_1}{\text{ch}_0}$ -length of $L_{EF} \cap \bar{\Delta}_{\leq 0}$ is the diameter of the semicircular potential wall of E and F . This is a measure of the size of the wall, and we have the following result, which says for walls of small length, there exists no stable object.

Corollary 1.32. *Let $w \in K(\mathbf{P}^2)$ be a non-zero character not inside the Le Potier cone Cone_{LP} , and σ be a geometric stability condition inside the cone $\bar{\Delta}_{<0}$. When the $\frac{\text{ch}_1}{\text{ch}_0}$ -length of $L_{w\sigma} \cap \bar{\Delta}_{\leq 0}$ is less than or equal to 1, there is no σ -stable object F of character w .*

Proof. We show the case when $\text{ch}_0(w) \geq 0$, the other case can be proved similarly. Among all integers $k \leq \frac{\text{ch}_1(w)}{\text{ch}_0(w)}$, let c be the largest one such that w is strictly above the line $L_{O(c-1)O(c)}$. Note that $O(c+1)^+$ is on the line $L_{O(c-1)O(c)}$, since w is not inside the Le Potier cone, we have $\frac{\text{ch}_1(w)}{\text{ch}_0(w)} \geq c+1$. Now w is not above the line $L_{O(c)O(c+1)}$, so the segment $L_{O(c+1)w} \cap \bar{\Delta}_{\leq 0}$ has $\frac{\text{ch}_1}{\text{ch}_0}$ -length greater than or equal to that of $L_{O(c)O(c+1)}$, which is 1. By assumption, σ is inside the cone $\bar{\Delta}_{<0}$, it must lie on or below the line $L_{O(c+1)w}$ and to the left of $L_{O(c+1)\pm}$. Note that $L_{O(c-1)O(c)}$ is just $L_{O(c+1)^+O(c)^+}$, by Proposition 1.27, there is no σ -stable object of character w . \square

Remark 1.33. *If F is σ -stable, in the proof we can see that below $L_{\sigma F}$ there exist the characters of at least two line bundles $O(c-1)$ and $O(c)$.*

2 Wall-crossing and canonical line bundles

In this section, we prove our first main theorem: the wall crossing in stability condition space induces the MMP for moduli of sheaves on \mathbf{P}^2 . In Section 2.1, we review the construction of moduli space of semistable objects as moduli of quiver representations. In Section 2.2, we prove the main technical result on vanishing of certain Ext^2 . In Section

2.1 Construction of the moduli space

2.3, the generic stability of extension objects is proved. This will be used in the proof of the irreducibility of moduli of stable objects, which occupies Section 2.4. We rephrase some results from variation of GIT in our situation in Section 2.5, and use this to prove our main theorem in Section 2.6.

2.1 Construction of the moduli space

In this section, we review the construction of the moduli space of σ -semistable objects on \mathbf{P}^2 with a given character via the geometric invariant theory. Let w be a Chern character and $\sigma_{s,q}$ be a geometric stability condition, we write $\mathfrak{M}_{\sigma_{s,q}}^{ss}(w)$ for the moduli space of $\sigma_{s,q}$ -(semi)stable objects in $\text{Coh}_{\#s}$ with character w . The line $L_{w\sigma_{s,q}}$ passes through $\text{MZ}_{\mathcal{E}}$ for some exceptional triple \mathcal{E} . We may choose a point P in $\text{MZ}_{\mathcal{E}}$ for some \mathcal{E} such that the line segment $l_{P\sigma_{s,q}}$ is contained in Geo_{LP} . By Corollary 1.21, the moduli space $\mathfrak{M}_{\sigma_{s,q}}^{ss}(w)$ is the same as $\mathfrak{M}_{\sigma_P}^{ss}(w)$.

Let \mathcal{E} be the exceptional triple consisting of E_1, E_2 and E_3 , and let $\mathcal{A}_{\mathcal{E}}$ be the heart $\langle E_1[2], E_2[1], E_3 \rangle$. We write the phase $\phi_P(E_i)$ of E_i at σ_P as ϕ_i . By Proposition 1.26, $\phi_1 < \phi_2 < \phi_3$ and $-1 < \phi_1 < \phi_3 - 1 < 0$. There is a real number t , $0 < t < 1$, such that $-2 < \phi_1 - t < -1 < \phi_2 - t < 0 < \phi_3 - t < 1$. Let the heart $\text{Coh}_P[t]$ be generated by σ_P -stable objects with phase in $(t, t+1]$, then it contains σ_P -stable objects $E_1[2], E_2[1]$ and E_3 . By Lemma 3.16 in [Ma], $\text{Coh}_P[t] = \mathcal{A}_{\mathcal{E}}$. For any σ_P -stable object F in Coh_P of character w , the phase $\phi_P(F)$ only depends on w , and is denoted by $\phi_P(w)$. When $\phi_P(w) - t > 0$, F is an object in $\mathcal{A}_{\mathcal{E}}$. In particular, when F is a coherent sheaf, there is a ‘resolution’ for F given as

$$0 \rightarrow E_1^{\oplus n_1} \rightarrow E_2^{\oplus n_2} \rightarrow E_3^{\oplus n_3} \rightarrow F \rightarrow 0.$$

The character $\vec{n} = (n_1, n_2, n_3)$ is the unique triple such that $n_1\tilde{v}(E_1) - n_2\tilde{v}(E_2) + n_3\tilde{v}(E_3) = w$. When $\phi_P(w) - t \leq 0$, $F[1]$ is an object in $\mathcal{A}_{\mathcal{E}}$. When F is a coherent sheaf, it appears as the cohomological sheaf at the middle term of

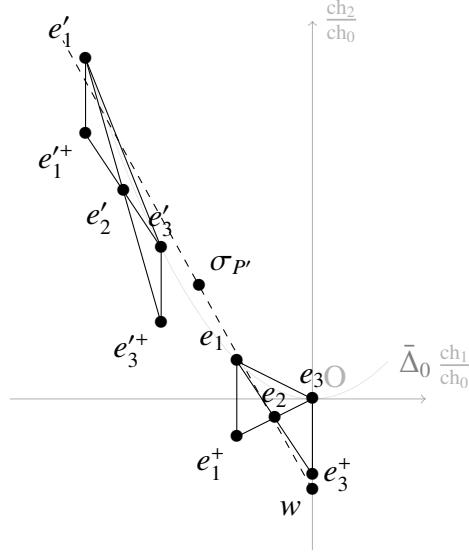
$$E_1^{\oplus n_1} \rightarrow E_2^{\oplus n_2} \rightarrow E_3^{\oplus n_3}.$$

The character $\vec{n} = (n_1, n_2, n_3)$ is the unique triple such that $n_1\tilde{v}(E_1) - n_2\tilde{v}(E_2) + n_3\tilde{v}(E_3) = -w$. The following easy lemma is useful to determine whether F or $F[1]$ is in $\mathcal{A}_{\mathcal{E}}$.

Lemma 2.1. *Let P be a point in $\text{MZ}_{\mathcal{E}}$, and F be a σ_P -stable object in $\text{Coh}_{\#P}$. If L_{PF} is above e_3 , then F is in the heart $\mathcal{A}_{\mathcal{E}}$. If L_{PF} is above e_1 , then $F[1]$ is in the heart $\mathcal{A}_{\mathcal{E}}$.*

Proof. By Lemma 1.17, when L_{PF} is above e_3 , we have the inequality $\phi_P(F) \geq \phi_P(E_3)$. Therefore, $\phi_P(F) - t > 0$ and F is in $\mathcal{A}_{\mathcal{E}}$. When L_{PF} is above e_1 , we have $\phi_P(F) < \phi_P(E_1[1])$. Therefore, $\phi_P(F) - t < \phi_P(E_1[1]) - t < 0$ and F is in $\mathcal{A}_{\mathcal{E}}[-1]$. \square

2.1 Construction of the moduli space



Picture: $F[1]$ is in $\mathcal{A}_{\mathcal{E}}$ and F is in $\mathcal{A}_{\mathcal{E}'}$

Remark 2.2. *The case when P is in $\text{TR}_{\mathcal{E}}$ and L_{PF} is below both e_1 and e_3 seems to be missing from the lemma. However, in this case, by Proposition 1.27, F is not σ_P -stable.*

We define $Q_{\mathcal{E}} = (Q_0, Q_1)$ to be the quiver associated to the exceptional triple \mathcal{E} . The set Q_0 has three vertices v_1, v_2 and v_3 . The arrow set Q_1 consists of $\text{hom}(E_1, E_2)$ arrows from v_1 to v_2 and $\text{hom}(E_2, E_3)$ arrows from v_2 to v_3 . Let $\vec{n} = (n_1, n_2, n_3)$ be a dimension character for $Q_{\mathcal{E}}$, and H_k be a complex linear space of dimension k , then the representation space $\mathbf{Rep}(Q_{\mathcal{E}}, \vec{n})$ can be identified with

$$\{(I, J) | I \in \text{Hom}(H_{n_1}, H_{n_2}) \otimes \text{Hom}(E_1, E_2), J \in \text{Hom}(H_{n_2}, H_{n_3}) \otimes \text{Hom}(E_2, E_3)\}.$$

We denote the composition map between E_i 's by $\alpha_{\mathcal{E}}$:

$$\alpha_{\mathcal{E}} : \text{Hom}(E_1, E_2) \otimes \text{Hom}(E_2, E_3) \rightarrow \text{Hom}(E_1, E_3).$$

This gives a relation of the quiver $Q_{\mathcal{E}}$ and we have the space of quiver representations with relation:

$$\mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}, \alpha_{\mathcal{E}}) := \{(I, J) \in \mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}) \mid J \circ I \in \text{Hom}(H_{n_1}, H_{n_3}) \otimes \ker \alpha_{\mathcal{E}}\}.$$

As a subvariety of $\mathbf{Rep}(Q_{\mathcal{E}}, \vec{n})$, $\mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}, \alpha_{\mathcal{E}})$ is determined by $JI = 0$, which contains $n_1 n_3 \text{hom}(E_1, E_3)$ equations.

The category $\mathcal{A}_{\mathcal{E}}$ is equivalent to the category of finite dimensional modules over the path algebra $(Q_{\mathcal{E}}, \alpha_{\mathcal{E}})$. Any object F in $\mathcal{A}_{\mathcal{E}}$ with character $n_1 \tilde{v}(E_1) - n_2 \tilde{v}(E_2) + n_3 \tilde{v}(E_3)$ can be written as a representation \mathbf{K}_F (unique up to the $G_{\vec{n}}$ -action) in $\mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}, \alpha_{\mathcal{E}})$.

Definition 2.3. *Let $\mathbf{K} = (I, J)$ and $\mathbf{K}' = (I', J')$ be two objects in $\mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}, \alpha_{\mathcal{E}})$ and $\mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}', \alpha_{\mathcal{E}})$, respectively. We introduce notations for the following sets of homomorphisms.*

$$\text{Hom}^i(\mathbf{K}, \mathbf{K}') := \bigoplus_j \text{Hom}_O(H_{n_j} \otimes E_j, H_{n'_{j+i}} \otimes E_{j+i}).$$

2.1 Construction of the moduli space

Here H_{n_i} and $H_{n'_i}$ are defined to be the zero space when $i \neq 0, 1, 2$. The derivatives d^0 and d^1 are linear maps defined as follows:

$$\begin{aligned} d^0 : \text{Hom}^0(\mathbf{K}, \mathbf{K}') &\rightarrow \text{Hom}^1(\mathbf{K}, \mathbf{K}') \\ (f_0, f_1, f_2) &\mapsto (I' \circ f_0 - f_1 \circ I, J' \circ f_1 - f_2 \circ J) \\ d^1 : \text{Hom}^1(\mathbf{K}, \mathbf{K}') &\rightarrow \text{Hom}^2(\mathbf{K}, \mathbf{K}') \\ (g_1, g_2) &\mapsto (J' \circ g_1 + g_2 \circ I). \end{aligned}$$

Let F and G be two objects in $\mathcal{A}_\mathcal{E}$, \mathbf{K}_F and \mathbf{K}_G be their representations in $\mathbf{Rep}(Q_\mathcal{E}, \alpha_\mathcal{E})$. The Ext^i groups of F and G can be computed via \mathbf{K}_F and \mathbf{K}_G .

Lemma 2.4. *The $\text{Ext}^*(F, G)$ groups are the cohomology of the complex*

$$\text{Hom}^0(\mathbf{K}_F, \mathbf{K}_G) \xrightarrow{d^0} \text{Hom}^1(\mathbf{K}_F, \mathbf{K}_G) \xrightarrow{d^1} \text{Hom}^2(\mathbf{K}_F, \mathbf{K}_G).$$

In particular,

$$\begin{aligned} \ker d^0 &\simeq \text{Hom}(F, G) \\ \text{Hom}^2(\mathbf{K}_F, \mathbf{K}_G)/\text{im } d^1 &\simeq \text{Ext}^2(F, G). \end{aligned}$$

Let $\vec{\rho}$ be a weight character for objects in $\mathbf{Rep}(Q_\mathcal{E}, \vec{n})$, in particular, $\vec{n} \cdot \vec{\rho} = 0$. An object \mathbf{K} in $\text{Rep}(Q, \vec{n})$ is $\vec{\rho}$ -(semi)stable if and only if for any non-zero proper sub-representation \mathbf{K}' of \mathbf{K} with dimension character $\vec{n}' < \vec{n}$, we have $\vec{n}' \cdot \vec{\rho} < (\leq) 0$.

Now we want to relate Bridgeland stability of objects to King stability of quiver representations. Let L be a line on the $\{1, \frac{\text{ch}_1}{\text{ch}_0}, \frac{\text{ch}_2}{\text{ch}_0}\}$ -plane not at the infinity. Suppose L intersects $l_{e_1 e_3}$ for an exceptional triple $\mathcal{E} (= \{E_1, E_2, E_3\})$. Let f be a linear function with variables ch_0, ch_1 and ch_2 such that the zero locus of f is L . Moreover we assume that $f(\tilde{v}(E_1))$ is positive. The weight character $\vec{\rho}_{L, \mathcal{E}}$ is given by

$$(f(\tilde{v}(E_1)), -f(\tilde{v}(E_2)), f(\tilde{v}(E_3)))$$

up to a positive scalar.

Lemma 2.5. *Let F be an object in $\mathcal{A}_\mathcal{E}$ and P be a point in $\text{MZ}_\mathcal{E}$ such that L_{FP} intersects $l_{e_1 e_3}$, then F (or $F[-1]$) is σ_P -(semi)stable if and only if \mathbf{K}_F is $\vec{\rho}_{L_{FP}, \mathcal{E}}$ -(semi)stable.*

Proof. First we want to modify the stability condition in a way that the central charge of the exceptional bundles are better behaved, and the weight character remains the same. Since L_{FP} intersects $l_{e_1 e_3}$, by Corollary 1.21, we may assume P is in the triangle area $\text{TR}_\mathcal{E}$. The central charges of objects $E_1[2]$, $E_2[1]$ and E_3 are

$$\begin{aligned} Z_P(E_1[2]) &= -\text{ch}_2(E_1) + q \text{ch}_0(E_1) + (\text{ch}_1(E_1) - s \text{ch}_0(E_1))i; \\ Z_P(E_2[1]) &= \text{ch}_2(E_2) - q \text{ch}_0(E_2) - (\text{ch}_1(E_2) - s \text{ch}_0(E_2))i; \\ Z_P(E_3) &= -\text{ch}_2(E_3) + q \text{ch}_0(E_3) + (\text{ch}_1(E_3) - s \text{ch}_0(E_3))i. \end{aligned}$$

2.1 Construction of the moduli space

There is a suitable real number $0 < t < 1$ such that the new central charge $Z_P^\bullet := e^{i\pi t} Z_P$ maps $E_1[2]$, $E_2[1]$ and E_3 to the upper half plane in \mathbb{C} .

Now we can rewrite the stability condition in terms of the weight character. Write $\vec{Z}_P^\bullet := (Z_P^\bullet(E_1[2]), Z_P^\bullet(E_2[1]), Z_P^\bullet(E_3)) = \vec{d}^\bullet + \vec{b}^\bullet i$ for two real vectors \vec{d}^\bullet and \vec{b}^\bullet . The object F is Z_P^\bullet -(semi)stable if and only if for any non-zero proper subobject F' in \mathcal{A}_ε ,

$$\text{Arg } Z_P^\bullet(F') < (\leq) \text{Arg } Z_P^\bullet(F).$$

In other words, suppose the dimension vector of \mathbf{K}_F is $\vec{n} = (n_1, n_2, n_3)$, then for any nonzero proper sub-representation $\mathbf{K}_{F'}$ with dimension vector \vec{n}' ,

$$\text{Arg } \vec{n}' \cdot \vec{Z}_P^\bullet < (\leq) \text{Arg } \vec{n} \cdot \vec{Z}_P^\bullet. \quad (1)$$

Let $\vec{\rho}^\bullet$ be the vector

$$-(\vec{b}^\bullet \cdot \vec{n})\vec{d}^\bullet + (\vec{d}^\bullet \cdot \vec{n})\vec{b}^\bullet.$$

Since each factor of \vec{b}^\bullet is non-negative, the inequality (1) holds if and only if

$$\frac{\vec{n} \cdot \vec{d}^\bullet}{\vec{n} \cdot \vec{b}^\bullet} < (\leq) \frac{\vec{n}' \cdot \vec{d}^\bullet}{\vec{n}' \cdot \vec{b}^\bullet}$$

if and only if $\vec{n}' \cdot \vec{\rho}^\bullet < (\leq) 0$.

We can also write $\vec{Z}_P := (Z_P(E_1[2]), Z_P(E_2[1]), Z_P(E_3)) = \vec{d} + \vec{b}i$ for two real vectors \vec{d} and \vec{b} . As $Z_P^\bullet = e^{i\pi t} Z_P$, the character $\vec{\rho} := (\vec{b} \cdot \vec{n})\vec{d} - (\vec{d} \cdot \vec{n})\vec{b}$ is the same as $\vec{\rho}^\bullet$.

At last we need to show that $\vec{\rho}$ is $\vec{\rho}_{L_{FP}, \varepsilon}$ up to a positive scalar. Let f be the linear function:

$$f(\text{ch}_0, \text{ch}_1, \text{ch}_2) := (\vec{d} \cdot \vec{n})(\text{ch}_1 - s \text{ch}_0) - (\vec{b} \cdot \vec{n})(-\text{ch}_2 + q \text{ch}_0).$$

The zero locus of f contains P because $f(1, s, q) = 0$. We also have

$$f(\tilde{v}(F)) = f(n_1 \tilde{v}(E_1) - n_2 \tilde{v}(E_2) + n_3 \tilde{v}(E_3)) = (\vec{d} \cdot \vec{n})(\vec{b} \cdot \vec{d}) - (\vec{b} \cdot \vec{n})(\vec{d} \cdot \vec{b}) = 0.$$

Therefore, the zero locus of f also contains $v(F)$.

It is easy to check that

$$(f(v(E_1)), -f(v(E_2)), f(v(E_3)))$$

is the vector $\vec{\rho}$. Since P is above the line $L_{E_1 E_2}$, $\phi_P(E_2[1]) < \phi_P(E_1[2])$ and the determinant $\det \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} < 0$. Similarly, $\det \begin{bmatrix} a_1 & a_3 \\ b_1 & b_3 \end{bmatrix} < 0$. Therefore, the first factor of $\vec{\rho}^\bullet$, which is $-\det \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} n_2 - \det \begin{bmatrix} a_1 & a_3 \\ b_1 & b_3 \end{bmatrix} n_3$, is always positive. So $f(v(E_1)) > 0$.

Now f satisfies the desired properties, and induces the weight character $\vec{\rho}$. Any other f satisfying the same properties induces the same character up to a positive scalar. \square

Remark 2.6. By the construction, the character $\vec{\rho}_{L_{FP}, \varepsilon}$ (up to a positive scalar) only depends on the wall L but not the position of P .

2.1 Construction of the moduli space

To conclude the construction of the moduli space of σ -stable objects via the geometric invariant theory, we summarize the previous notations as follows. Let w be a character and σ_P a geometric stability condition. Suppose P is in $\text{MZ}_{\mathcal{E}}$ for some exceptional triple $\mathcal{E} = \langle E_1, E_2, E_3 \rangle$ such that L_{Pw} intersects $l_{E_1 E_3}$. We assume that w or $-w$ can be written as $n_1 \tilde{v}(E_1) - n_2 \tilde{v}(E_2) + n_3 \tilde{v}(E_3)$ for a positive dimension character $\vec{n}_w = (n_1, n_2, n_3)$. Let $G_{\vec{n}}$ be the group $\text{GL}(H_{n_1}) \times \text{GL}(H_{n_2}) \times \text{GL}(H_{n_3})$ acting naturally on the space of $\mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}_w, \alpha_{\mathcal{E}})$ and $\mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}_w)$ with stabilizer containing the scalar group \mathbb{C}^{\times} .

Proposition 2.7. *Adopt the notation as above, the moduli space $\mathfrak{M}_{\sigma}^{ss}(w)$ or $\mathfrak{M}_{\sigma}^{ss}(-w)$ of σ_P -semistable objects in $\text{Coh}_{\sharp P}$ can be constructed as the GIT quotient space*

$$\begin{aligned} & \mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}_w, \alpha_{\mathcal{E}}) //_{\det^{\vec{\rho}_{L_{w\sigma}, \mathcal{E}}}} (G_{\vec{n}_w} / \mathbb{C}^{\times}) \\ &= \mathbf{Proj} \bigoplus_{m \geq 0} \mathbb{C}[\mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}_w, \alpha_{\mathcal{E}})]^{G_{\vec{n}_w} / \mathbb{C}^{\times}, \det^{m\vec{\rho}_{L_{w\sigma}, \mathcal{E}}}}. \end{aligned}$$

Proof. By the previous discussion and Lemma 2.5, the moduli space $\mathfrak{M}_{\sigma}^{ss}(w)$ or $\mathfrak{M}_{\sigma}^{ss}(-w)$ parameterizes the $\vec{\rho}_{L_{w\sigma}, \mathcal{E}}$ -semistable objects in $\mathcal{A}_{\mathcal{E}}$ with dimension character \vec{n}_w . By King's criterion, Proposition 3.1 in [Ki], \mathbf{K} in $\mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}_w, \alpha_w)$ is $\vec{\rho}_{L_{w\sigma}}$ -semistable if the point of \mathbf{K} in the space $\mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}_w)$ is $\det^{\vec{\rho}_{L_{w\sigma}}}$ -semistable with respect to the $G_{\vec{n}_w} / \mathbb{C}^{\times}$ -action. The map

$$\begin{aligned} \mathbb{C}[\mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}_w)]^{G, \det^{m\vec{\rho}}} &\rightarrow \left(\mathbb{C}[\mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}_w)] / I_{\alpha_{\mathcal{E}}} \right)^{G, \det^{m\vec{\rho}}} \\ &= \mathbb{C}[\mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}_w, \alpha_w)]^{G, \det^{m\vec{\rho}}}. \end{aligned}$$

is surjective because the group $G = G_{\vec{n}_w} / \mathbb{C}^{\times}$ is semisimple. Therefore, the relation $\alpha_{\mathcal{E}}$ does not affect the stability condition. In other words, a point \mathbf{K} is $\det^{\vec{\rho}_{L_{w\sigma}}}$ -semistable with respect to the $G_{\vec{n}_w} / \mathbb{C}^{\times}$ -action on the space $\mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}_w)$ if and only if on the space $\mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}, \alpha_{\mathcal{E}})$. As explained in Chapter 2.2 in [Gi] by Ginzburg, the moduli space is constructed as the GIT quotient in the proposition. \square

Now we have the following consequence on the finiteness of actual walls.

Proposition 2.8. *1. Let w be a character in $\mathbf{K}(\mathbf{P}^2)$, then there are only finitely many actual walls for $\mathfrak{M}_{\sigma}^{ss}(w)$.*

2. Suppose $\text{ch}_0(w) > 0$, then for any $s < \frac{\text{ch}_1}{\text{ch}_0}(w)$ and q large enough (depending on s), the moduli space $\mathfrak{M}_{\sigma_{s,q}}^{ss}(w)$ is the same as the moduli space $\mathfrak{M}_{GM}^{ss}(w)$ of Gieseker (semi)stable coherent sheaves.

Proof. By Corollary 1.32, we only need to consider the region from the vertical wall to the tangent line of $\tilde{\Delta}_0$. We may choose finitely many quiver region $\text{MZ}_{\mathcal{E}}$ such that each ray from w contained in this region passes through at least one $\text{MZ}_{\mathcal{E}}$. In each $\text{MZ}_{\mathcal{E}}$, there are finitely many walls, because there are only finitely many dimension vectors of possible destabilizing subobjects.

The second statement is a consequence of the first statement and the standard fact that $\sigma_{s,q}$ tends to Gieseker stable condition when q tends to infinity. \square

2.2 The Ext^2 vanishing property

2.2 The Ext^2 vanishing property

In this section we prove the most important technical lemma. It is about the vanishing property of Ext^2 of σ -stable objects. This property is trivial in the slope stable situation by Serre duality. But it is more involved in the Bridgeland stability situation since the objects may not be in the same heart.

Lemma 2.9. *Let σ_P be a geometric stability condition with P in $\bar{\Delta}_{<0}$, E and F be two σ_P -stable objects in $\text{Coh}_{\#P}$. Suppose P , $v(E)$ and $v(F)$ are collinear; then*

$$\text{Hom}(E, F[2]) = \text{Hom}(F, E[2]) = 0.$$

Proof. Case 1: At least one of $\tilde{v}(E)$ and $\tilde{v}(F)$ is an exceptional character. Assume that $\tilde{v}(E)$ is exceptional. Suppose the dyadic number corresponding to $\tilde{v}(E)$ is $\frac{p}{2^q}$, let E_1 and E_3 be exceptional bundles corresponding to dyadic numbers $\frac{p-1}{2^q}$ and $\frac{p+1}{2^q}$ respectively. Then $\text{Hom}(E_1, E)$ and $\text{Hom}(E, E_3)$ are both non-zero. As E is σ -stable, $l_{E\sigma}$ does not intersect $l_{e_1e_1^+}$ nor $l_{e_3e_3^+}$, otherwise this contradicts to Lemma 1.17. We may assume that σ is in $\text{MZ}_{\mathcal{E}}$, where $\mathcal{E} = \{E_1, E, E_3\}$. As F has the same phase with E at σ , $F[1]$ is in $\mathcal{A}_{\mathcal{E}}$. Since

$$\text{Hom}(E[1], E_1[2+s]) = 0 \text{ for all } s \in \mathbb{Z};$$

$$\text{Hom}(E[1], E[1+s]) = 0 \text{ for all } s \neq 0;$$

$$\text{Hom}(E[1], E_3[s]) = 0 \text{ for all } s \neq 1,$$

$\text{Hom}(E[1], G[s]) = 0$, for any object G in $\mathcal{A}_{\mathcal{E}}$ when $s \neq 0$ or 1 . Therefore, $\text{Hom}(E, F[2]) = \text{Hom}(E[1], F[1+2]) = 0$. Similarly, We have $\text{Hom}(G, E[1+s]) = 0$, for any object G in $\mathcal{A}_{\mathcal{E}}$ when $s \neq 0$ or 1 . Therefore, $\text{Hom}(F, E[2]) = 0$.

Case 2: Neither $\tilde{v}(E)$ nor $\tilde{v}(F)$ is exceptional. By Corollary 1.30, their corresponding points are below the Le Potier curve C_{LP} .

Case 2.1: The $\frac{\text{ch}_1}{\text{ch}_0}$ -length of $L_{EF} \cap \bar{\Delta}_{\leq 0}$ is greater than 3.

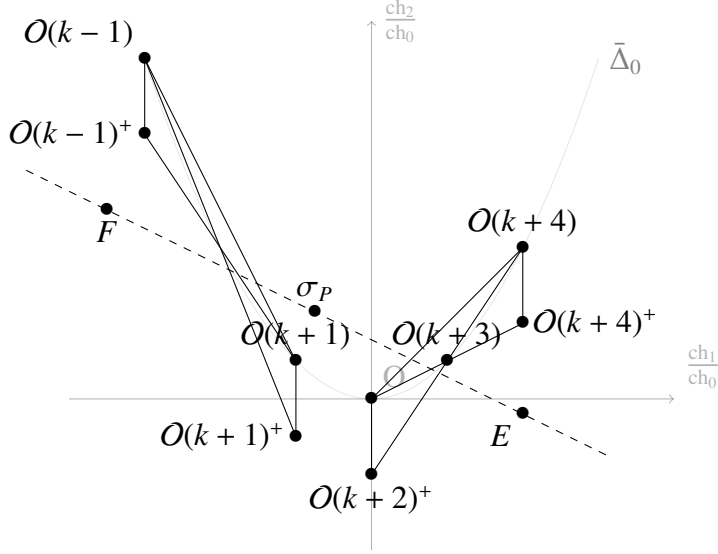
By the construction of Bridgeland stability conditions, it is easy to see that the objects $E(-3)$ and $F(-3)$ are also stable for any geometric stability conditions on the line $L_{E(-3)F(-3)}$. Since the $\frac{\text{ch}_1}{\text{ch}_0}$ -length of $L_{EF} \cap \bar{\Delta}_{\leq 0}$ is greater than 3, the intersection point Q of L_{EF} and $L_{E(-3)F(-3)}$ is in $\bar{\Delta}_{\leq 0}$. By Corollary 1.21, the objects E , F , $E(-3)$ and $F(-3)$ are all σ_Q -stable. By Lemma 1.17, $\phi_Q(E(-3)) = \phi_Q(F(-3)) < \phi_Q(E) = \phi_Q(F)$. We have

$$\text{Hom}(E, F(-3)), \text{Hom}(F, E(-3)) = 0.$$

The statement then holds by Serre duality.

Case 2.2: The $\frac{\text{ch}_1}{\text{ch}_0}$ -length of $L_{EF} \cap \bar{\Delta}_{\leq 0}$ is not greater than 3.

2.2 The Ext^2 vanishing property



Picture: $L_{EF} \cap \bar{\Delta}_{\leq 0}$ is not greater than 3.

Since E is σ_P -stable, by Corollary 1.32 and its proof, there exists an integer k such that the points $v(O(k+1))$, which is $(1, k+1, \frac{(k+1)^2}{2})$ and $v(O(k+2))$ are below the segment $L_{EF} \cap \bar{\Delta}_{\leq 0}$ (see above picture, where e_1, e_2, e_3 correspond to $O(k+2), O(k+3), O(k+4)$ respectively). Equivalently, the points $v(O(k-1))$ and $v(O(k-2))$ are below the segments $L_{E(-3)F(-3)} \cap \bar{\Delta}_{\leq 0}$.

Let \mathcal{E}_k be the exceptional triple $\langle O(k-1), O(k), O(k+1) \rangle$, then by our assumption, $L_{O(k-1)O(k+1)}$ must intersect both L_{EF} and $L_{E(-3)F(-3)}$. By Lemma 2.1, as the point $v(O(k+1))$ is below the segment $L_{EF} \cap \bar{\Delta}_{\leq 0}$, E and F are both in $\mathcal{A}_{\mathcal{E}_k}$. As the point $v(O(k-1))$ is below the segment $L_{E(-3)F(-3)} \cap \bar{\Delta}_{\leq 0}$, both $E(-3)[1]$ and $F(-3)[1]$ are in $\mathcal{A}_{\mathcal{E}_k}$. Therefore, we have

$$\text{Hom}(E, F(-3)) = \text{Hom}(E, (F(-3)[1])[-1]) = 0.$$

By Serre duality, the statement holds. \square

In particular, if an object E is σ -semistable for some geometric stability condition σ , we have $\text{Hom}(E, E[2]) = 0$. To see this, E admits σ -stable Jordan-Holder filtrations, and for any two stable factors we have the $\text{Hom}(-, -[2])$ vanishing, hence $\text{Hom}(E, E[2]) = 0$.

As an immediate application, $\mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}, \alpha_{\mathcal{E}})^{\vec{\beta}-ss}$ is smooth.

Corollary 2.10. *Let x be a point in $\mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}_w, \alpha_{\mathcal{E}})^{\vec{\beta}-ss}$, then as a closed subvariety of $\mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}_w)$, $\mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}_w, \alpha_{\mathcal{E}})$ is smooth at the point x .*

Proof. Let $\mathbf{K} = (I_0, J_0)$ be the quiver representation that x stands for. The dimension of the Zariski tangent space at x is the dimension of

$$\text{Hom}_{\mathbb{C}}(\mathbb{C}[\mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}_w)]/(J \circ I), \mathbb{C}[t]/(t^2))$$

at (I_0, J_0) . Each tangent direction can be written in the form $(I_0, J_0) + t(I_1, J_1)$. In order to satisfy the equation $J \circ I \in (t^2)$, we need

$$J_0 \circ I_1 + J_1 \circ I_0 = 0.$$

2.3 Generic stability

Hence the space of (I_1, J_1) is just the kernel of $d^1 : \text{Hom}^1(\mathbf{K}, \mathbf{K}) \rightarrow \text{Hom}^2(\mathbf{K}, \mathbf{K})$. By Lemma 2.9, d^1 is surjective. The Zariski tangent space has dimension $\text{hom}^1(\mathbf{K}, \mathbf{K}) - \text{hom}^2(\mathbf{K}, \mathbf{K})$. On the other hand, $\mathbf{Rep}(Q_\mathcal{E}, \vec{n}, \alpha_\mathcal{E})$ is the zero locus of $n_1 n_3 \cdot \text{hom}(E_1, E_3) = \text{hom}^2(\mathbf{K}, \mathbf{K})$ equations, hence each irreducible component is of dimension at least $\text{hom}^1(\mathbf{K}, \mathbf{K}) - \text{hom}^2(\mathbf{K}, \mathbf{K})$, which is not less than the dimension of the Zariski tangent space at x . Therefore, $\mathbf{Rep}(Q_\mathcal{E}, \vec{n}_w, \alpha_\mathcal{E})$ is smooth at the point x . \square

Remark 2.11. When the dimension character \vec{n} is primitive, $G(= G_{\vec{n}_w}/\mathbb{C}^\times)$ acts freely on the stable locus. By Luna's étale slice theorem,

$$\mathbf{Rep}(Q_\mathcal{E}, \vec{n}_w, \alpha_\mathcal{E})^{\vec{\rho}-s} \rightarrow \mathbf{Rep}(Q_\mathcal{E}, \vec{n}_w, \alpha_\mathcal{E})^{\vec{\rho}-s}/G$$

is a principal G -bundle. Since $\mathbf{Rep}(Q_\mathcal{E}, \vec{n}_w, \alpha_\mathcal{E})^{\vec{\rho}-s}$ is smooth, by Proposition IV.17.7.7 in [Gr], the base space is also smooth.

2.3 Generic stability

Based on Lemma 2.9, we establish some estimate on the dimension of strictly semistable objects in this section. The technical result Lemma 2.13 is useful in the proof for the irreducibility of the moduli space.

Definition 2.12. Suppose $\vec{n} = \vec{n}' + \vec{n}''$ such that $\vec{n}' \cdot \vec{\rho} = \vec{n}'' \cdot \vec{\rho} = 0$. Choose $\mathbf{F} \in \mathbf{Rep}(Q_\mathcal{E}, \vec{n}', \alpha_\mathcal{E})^{\vec{\rho}-ss}$ and $\mathbf{G} \in \mathbf{Rep}(Q_\mathcal{E}, \vec{n}'', \alpha_\mathcal{E})^{\vec{\rho}-ss}$. We write $\mathbf{Rep}(Q_\mathcal{E}, \mathbf{F}, \mathbf{G})$ as the subspace in $\mathbf{Rep}(Q_\mathcal{E}, \vec{n}, \alpha_\mathcal{E})^{\vec{\rho}-ss}$ consisting of representations \mathbf{K} that can be written as an extension of \mathbf{G} by \mathbf{F} :

$$0 \rightarrow \mathbf{F} \rightarrow \mathbf{K} \rightarrow \mathbf{G} \rightarrow 0.$$

We also write $\mathbf{Rep}(Q_\mathcal{E}, \vec{n}', \vec{n}'')$ for the union of all $\mathbf{Rep}(Q_\mathcal{E}, \mathbf{F}, \mathbf{G})$ such that $\mathbf{F} \in \mathbf{Rep}(Q_\mathcal{E}, \vec{n}', \alpha_\mathcal{E})^{\vec{\rho}-ss}$ and $\mathbf{G} \in \mathbf{Rep}(Q_\mathcal{E}, \vec{n}'', \alpha_\mathcal{E})^{\vec{\rho}-ss}$.

We have the following dimension estimate for $\mathbf{Rep}(Q_\mathcal{E}, \mathbf{F}, \mathbf{G})$:

Lemma 2.13.

$$\dim \mathbf{Rep}(Q_\mathcal{E}, \mathbf{F}, \mathbf{G}) \leq -\chi(\mathbf{G}, \mathbf{F}) + \dim G_{\vec{n}} - \text{hom}(\mathbf{F}, \mathbf{F}) - \text{hom}(\mathbf{G}, \mathbf{G}).$$

Proof. Let $X(\mathbf{F}, \mathbf{G})$ be the subset of $\mathbf{Rep}(Q_\mathcal{E}, \mathbf{F}, \mathbf{G})$ consisting of objects of the form:

$$I = \begin{pmatrix} I_{\mathbf{F}} & I(\mathbf{G}, \mathbf{F}) \\ 0 & I_{\mathbf{G}} \end{pmatrix}, J = \begin{pmatrix} J_{\mathbf{F}} & J(\mathbf{G}, \mathbf{F}) \\ 0 & J_{\mathbf{G}} \end{pmatrix},$$

for a pair $(I(\mathbf{G}, \mathbf{F}), J(\mathbf{G}, \mathbf{F})) \in \text{Hom}^1(\mathbf{G}, \mathbf{F})$. The morphisms are shown in the following diagram:

$$\begin{array}{ccccc} \mathbf{F} : & \mathbb{C}^{n'_1} & \xrightarrow{I_{\mathbf{F}}} & \mathbb{C}^{n'_2} & \xrightarrow{J_{\mathbf{F}}} & \mathbb{C}^{n'_3} \\ & & \nearrow I(\mathbf{G}, \mathbf{F}) & & \nearrow J(\mathbf{G}, \mathbf{F}) \\ \mathbf{G} : & \mathbb{C}^{n''_1} & \xrightarrow{I_{\mathbf{G}}} & \mathbb{C}^{n''_2} & \xrightarrow{J_{\mathbf{G}}} & \mathbb{C}^{n''_3} \end{array}$$

2.3 Generic stability

Due to the condition that $J \circ I \in \ker \alpha_{\mathcal{E}} \otimes \text{Hom}(\mathbb{C}^{n_1}, \mathbb{C}^{n_3})$, the pair $(I(\mathbf{G}, \mathbf{F}), J(\mathbf{G}, \mathbf{F}))$ is contained in the kernel of the morphism

$$d^1(\mathbf{G}, \mathbf{F}) : \text{Hom}^1(\mathbf{G}, \mathbf{F}) \rightarrow \text{Hom}^2(\mathbf{G}, \mathbf{F}).$$

By Lemma 2.9, $d^1(\mathbf{G}, \mathbf{F})$ is surjective, hence

$$\dim X(\mathbf{F}, \mathbf{G}) \leq \text{hom}^1(\mathbf{G}, \mathbf{F}) - \text{hom}^2(\mathbf{G}, \mathbf{F}).$$

Each element $g \in GL_{\vec{n}}$ can be written as a block matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where $A \in \text{Hom}^0(\mathbf{F}, \mathbf{F})$, $B \in \text{Hom}^0(\mathbf{G}, \mathbf{F})$, $C \in \text{Hom}^0(\mathbf{F}, \mathbf{G})$ and $D \in \text{Hom}^0(\mathbf{G}, \mathbf{G})$. Note that when $A \in \text{Hom}(\mathbf{F}, \mathbf{F})$, $D \in \text{Hom}(\mathbf{G}, \mathbf{G})$ and $C = 0$, we have $g \cdot X(\mathbf{F}, \mathbf{G}) = X(\mathbf{F}, \mathbf{G})$. Therefore,

$$\begin{aligned} \dim \mathbf{Rep}(Q_{\mathcal{E}}, \mathbf{F}, \mathbf{G}) &= \dim G_{\vec{n}} \cdot X(\mathbf{F}, \mathbf{G}) \\ &\leq \dim G_{\vec{n}} + \dim X(\mathbf{F}, \mathbf{G}) - \text{hom}(\mathbf{F}, \mathbf{F}) - \text{hom}(\mathbf{G}, \mathbf{G}) - \text{hom}^0(\mathbf{G}, \mathbf{F}) \\ &\leq -\chi(\mathbf{G}, \mathbf{F}) + \dim G_{\vec{n}} - \text{hom}(\mathbf{F}, \mathbf{F}) - \text{hom}(\mathbf{G}, \mathbf{G}). \end{aligned}$$

□

Definition 2.14. $\mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}, \alpha_{\mathcal{E}})^{\vec{\rho}-ss}_c := \{\mathbf{F} \in \mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}, \alpha_{\mathcal{E}})^{\vec{\rho}-ss} \mid \text{hom}(\mathbf{F}, \mathbf{F}) = c\}$.
 $\mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}', \vec{n}'')^{\vec{\rho}-ss}_{c,d} := \{\mathbf{K} \in \mathbf{Rep}(Q_{\mathcal{E}}, \mathbf{F}, \mathbf{G}) \mid \text{hom}(\mathbf{F}, \mathbf{F}) = c, \text{hom}(\mathbf{G}, \mathbf{G}) = d\}$.

The following proposition shows that given a Chern character w not inside Cone_{LP} and a generic stability condition σ , stable objects are dense in the moduli space $\mathfrak{M}_{\sigma}^{ss}(w)$. Note that this is a non-trivial statement only when w is not primitive.

Proposition 2.15. *Let \vec{n} be a character for $\mathbf{Rep}(Q_{\mathcal{E}}, \alpha_{\mathcal{E}})$ such that $\chi(\vec{n}, \vec{n}) \leq -1$. Let $\vec{\rho}$ be a generic weight with respect to \vec{n} , in other words, $\vec{\rho} \cdot \vec{n}' \neq 0$ for any $\vec{n}' < \vec{n}$ that is not proportional to \vec{n} . We have*

$$\dim \mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}, \alpha_{\mathcal{E}})^{\vec{\rho}-ss} = -\chi(\vec{n}, \vec{n}) + \dim G_{\vec{n}},$$

for each irreducible component of $\mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}, \alpha_{\mathcal{E}})^{\vec{\rho}-ss}$. Moreover,

$$\dim(\mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}, \alpha_{\mathcal{E}})^{\vec{\rho}-ss} \setminus \mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}, \alpha_{\mathcal{E}})^{\vec{\rho}-s}) \leq -\chi(\vec{n}, \vec{n}) + \dim G_{\vec{n}} - 1.$$

In particular, there is no component whose objects are all strictly semistable objects.

Proof. The first statement basically follows from the proof of Corollary 2.10. Just note that $\text{hom}^1(\mathbf{K}, \mathbf{K}) - \text{hom}^2(\mathbf{K}, \mathbf{K})$ in that proof is exactly $-\chi(\vec{n}, \vec{n}) + \dim G_{\vec{n}}$ here. We will repeat the proof:

$\mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}, \alpha_{\mathcal{E}})$ is the zero locus of $n_1 n_3 \cdot \text{hom}(E_1, E_3)$ equations, hence each irreducible component is of dimension at least $-\chi(\vec{n}, \vec{n}) + \dim G_{\vec{n}}$.

On the other hand, for any $\vec{\rho}$ -semistable object $\mathbf{K} \in \mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}, \alpha_{\mathcal{E}})^{\vec{\rho}-ss}$, $d^1 : \text{Hom}^1(\mathbf{K}, \mathbf{K}) \rightarrow \text{Hom}^2(\mathbf{K}, \mathbf{K})$ is surjective by Lemma 2.9, the Zariski tangent space is of dimension $-\chi(\vec{n}, \vec{n}) + \dim G_{\vec{n}}$. Since $\mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}, \alpha_{\mathcal{E}})^{\vec{\rho}-ss}$ is open in $\mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}, \alpha_{\mathcal{E}})$, for each irreducible component of $\mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}, \alpha_{\mathcal{E}})^{\vec{\rho}-ss}$, its dimension is $-\chi(\vec{n}, \vec{n}) + \dim G_{\vec{n}}$.

2.4 The irreducibility of the moduli space

For the second statement, when \vec{n} is primitive and $\vec{\rho}$ is generic, we have

$$\mathbf{Rep}(Q_E, \vec{n}, \alpha_E)^{\vec{\rho}-ss} = \mathbf{Rep}(Q_E, \vec{n}, \alpha_E)^{\vec{\rho}-s},$$

so the statement holds automatically in this case.

We may assume that $\vec{n} = m\vec{n}_0$, in which \vec{n}_0 is primitive. Since $\vec{\rho}$ is generic, any strictly semistable object must be destabilized by an object in $\mathbf{Rep}(Q_E, a\vec{n}_0)^{\vec{\rho}-ss}$ for some $0 < a < m$. Hence

$$\mathbf{Rep}(Q_E, \vec{n}, \alpha_E)^{\vec{\rho}-ss} \setminus \mathbf{Rep}(Q_E, \vec{n}, \alpha_E)^{\vec{\rho}-s} = \bigcup_{1 \leq a \leq m-1} \mathbf{Rep}(Q_E, a\vec{n}_0, (m-a)\vec{n}_0).$$

For each object $\mathbf{F} \in \mathbf{Rep}(Q_E, a\vec{n}_0, \alpha_E)^{\vec{\rho}-ss}$, the orbit $G_{a\vec{n}_0} \cdot \mathbf{F}$ in $\mathbf{Rep}(Q_E, a\vec{n}_0, \alpha_E)^{\vec{\rho}-ss}$ is of dimension $\dim G_{a\vec{n}_0} - c$. Therefore, by Lemma 2.13, we have

$$\begin{aligned} & \dim \mathbf{Rep}(Q_E, a\vec{n}_0, (m-a)\vec{n}_0)^{\vec{\rho}-ss}_{c,d} \\ & \leq -\chi((m-a)\vec{n}_0, a\vec{n}_0) + \dim G_{\vec{n}} - c - d - (\dim G_{a\vec{n}_0} - c) - (\dim G_{(m-a)\vec{n}_0} - d) \\ & \quad + \dim \mathbf{Rep}(Q_E, a\vec{n}_0, \alpha_E)^{\vec{\rho}-ss}_c + \dim \mathbf{Rep}(Q_E, (m-a)\vec{n}_0, \alpha_E)^{\vec{\rho}-ss}_d \\ & \leq -\chi((m-a)\vec{n}_0, a\vec{n}_0) + \dim G_{\vec{n}} - \chi((m-a)\vec{n}_0, (m-a)\vec{n}_0) - \chi(a\vec{n}_0, a\vec{n}_0) \\ & = -\chi(\vec{n}, \vec{n}) + \dim G_{\vec{n}} + \chi(a\vec{n}_0, (m-a)\vec{n}_0) \\ & \leq -\chi(\vec{n}, \vec{n}) + \dim G_{\vec{n}} - 1. \end{aligned}$$

The last inequality holds since $\chi(\vec{n}, \vec{n}) \leq -1$. Therefore,

$$\begin{aligned} & \dim \left(\mathbf{Rep}(Q_E, \vec{n}, \alpha_E)^{\vec{\rho}-ss} \setminus \mathbf{Rep}(Q_E, \vec{n}, \alpha_E)^{\vec{\rho}-s} \right) \\ & \leq \max_{c,d} \left\{ \dim \mathbf{Rep}(Q_E, a\vec{n}_0, (m-a)\vec{n}_0)^{\vec{\rho}-ss}_{c,d} \right\} \\ & \leq -\chi(\vec{n}, \vec{n}) + \dim G_{\vec{n}} - 1. \end{aligned}$$

In particular, since each component is of dimension $-\chi(\vec{n}, \vec{n}) + \dim G_{\vec{n}}$, there is no component consisting of strictly semistable objects. \square

2.4 The irreducibility of the moduli space

Based on the results and methods in the previous sections, we are able to estimate the dimension of the space of new stable objects after a wall-crossing. When the wall is to the left of the vertical wall, we show that the new stable objects in the next chamber has codimension at least 3. Together with Proposition 2.15, this will imply the irreducibility of the moduli space.

Let w be a character in $K(\mathbf{P}^2)$ with $\text{ch}_0(w) \geq 0$ and $\sigma_{s,q}$ a stability condition with $s < \frac{\text{ch}_1(w)}{\text{ch}_0(w)}$ such that $(1, s, q)$ is contained in MZ_E . Let \vec{n} be the dimension character for w in Q_E , and $\vec{\rho}$ be the weight character corresponding to $L_{w\sigma}$. Let $\vec{\rho}_-$ be the character in the chamber below $L_{w\sigma}$ and $\vec{\rho}_+$ in the chamber above $L_{w\sigma}$. The following two lemmas will be used in the proof of Proposition 2.18.

2.4 The irreducibility of the moduli space

Lemma 2.16. *Suppose \mathbf{K} is an object in $\mathbf{Rep}(Q_\varepsilon, \vec{n}_w, \alpha_\varepsilon)^{\vec{\rho}_- - s} \setminus \mathbf{Rep}(Q_\varepsilon, \vec{n}_w, \alpha_\varepsilon)^{\vec{\rho}_+ - s}$, then it can be written as a non-trivial extension*

$$0 \rightarrow \mathbf{K}' \rightarrow \mathbf{K} \rightarrow \mathbf{K}'' \rightarrow 0$$

of objects in $\mathbf{Rep}(Q_\varepsilon, \alpha_\varepsilon)$ such that the dimension character \vec{n}' of \mathbf{K}' satisfies $\vec{\rho}_- \cdot \vec{n}' < 0 = \vec{\rho} \cdot \vec{n}'$, and $\text{Hom}(\mathbf{K}'', \mathbf{K}') = 0$.

Proof. By the assumption on \mathbf{K} , it is a strictly $\vec{\rho}$ -semistable object, and is destabilized by a non-zero $\vec{\rho}$ -stable proper subobject \mathbf{K}' with $\vec{\rho} \cdot \vec{n}' = 0$. As \mathbf{K} is $\vec{\rho}_-$ -stable, we have $\vec{\rho}_- \cdot \vec{n}' < 0$. Let the quotient be \mathbf{K}'' , then \mathbf{K}' and \mathbf{K}'' are the objects we want.

In order to see that $\text{Hom}(\mathbf{K}'', \mathbf{K}') = 0$, suppose there is a non-zero map in $\text{Hom}(\mathbf{K}'', \mathbf{K}')$, then its image $\tilde{\mathbf{K}}$ in \mathbf{K}' is both a sub-representation and quotient representation of \mathbf{K} . Let \vec{n} be the dimension vector of $\tilde{\mathbf{K}}$. As \mathbf{K} is $\vec{\rho}_-$ -stable, we get $\vec{\rho}_- \cdot \vec{n} < 0 < \vec{\rho}_- \cdot \vec{n}$, which leads to a contradiction. \square

For a dimension vector \vec{n} of Q_ε , we write $\text{ch}_i(\vec{n})$ for $n_1 \text{ch}_i(E_1) - n_2 \text{ch}_i(E_2) + n_3 \text{ch}_i(E_3)$, $i = 0, 1, 2$.

Lemma 2.17. *Let \vec{n} and \vec{m} be two dimension vectors of Q_ε .*

1. The Euler character $\chi(\vec{n}, \vec{m})$ can be computed as

$$\begin{aligned} & \text{ch}_2(\vec{n})\text{ch}_0(\vec{m}) + \text{ch}_2(\vec{m})\text{ch}_0(\vec{n}) - \text{ch}_1(\vec{n})\text{ch}_1(\vec{m}) \\ & + \frac{3}{2}(\text{ch}_1(\vec{m})\text{ch}_0(\vec{n}) - \text{ch}_0(\vec{m})\text{ch}_1(\vec{n})) + \text{ch}_0(\vec{n})\text{ch}_0(\vec{m}). \end{aligned}$$

2. Suppose $\text{ch}_0(\vec{n}) \leq 0$, let w be of character $-(\text{ch}_0(\vec{n}), \text{ch}_1(\vec{n}), \text{ch}_2(\vec{n}))$ and P be a point in $\bar{\Delta}_{<0}$ to the left of the vertical wall $L_{w\pm}$ such that L_{Pw} intersects $l_{E_1 E_3}$. Let $\vec{\rho}$ be $\vec{\rho}_{L_{Pw}}$, and $\vec{\rho}_-$ be the character in the chamber below L_{Pw} . Suppose \vec{m} satisfies $\vec{\rho}_- \cdot \vec{m} < 0 = \vec{\rho} \cdot \vec{m}$ and \vec{n} satisfies $\vec{\rho}_- \cdot \vec{n} = 0 = \vec{\rho} \cdot \vec{n}$, then

$$\text{ch}_0(\vec{n})\text{ch}_1(\vec{m}) - \text{ch}_0(\vec{m})\text{ch}_1(\vec{n}) > 0.$$

Proof. The first statement follows from the Hirzebruch-Riemann-Roch formula for \mathbf{P}^2 :

$$\begin{aligned} \chi(F, G) &= \text{ch}_2(F)\text{ch}_0(G) + \text{ch}_2(G)\text{ch}_0(F) - \text{ch}_1(F)\text{ch}_1(G) \\ &+ \frac{3}{2}(\text{ch}_1(G)\text{ch}_0(F) - \text{ch}_0(G)\text{ch}_1(F)) + \text{ch}_0(F)\text{ch}_0(G). \end{aligned}$$

For the second statement, by definition of $\vec{\rho}$, $\vec{\rho}_-$ is in the same chamber as $\vec{\rho} + \epsilon(0, n_3, -n_2)$ for small enough $\epsilon > 0$. We have

$$\text{ch}_0(\vec{n})\text{ch}_1(\vec{m}) - \text{ch}_0(\vec{m})\text{ch}_1(\vec{n}) = (m_1, m_2, m_3) \cdot \vec{\Upsilon},$$

where $\vec{\Upsilon}$ is the vector $\left(\begin{vmatrix} \text{ch}_0(\vec{n}) & \text{ch}_1(\vec{n}) \\ \text{ch}_0(E_1) & \text{ch}_1(E_1) \end{vmatrix}, -\begin{vmatrix} \text{ch}_0(\vec{n}) & \text{ch}_1(\vec{n}) \\ \text{ch}_0(E_2) & \text{ch}_1(E_2) \end{vmatrix}, \begin{vmatrix} \text{ch}_0(\vec{n}) & \text{ch}_1(\vec{n}) \\ \text{ch}_0(E_3) & \text{ch}_1(E_3) \end{vmatrix} \right)$. The vector $\vec{\Upsilon}$ is a weight character for \vec{n} since $\vec{n} \cdot \vec{\Upsilon} = \text{ch}_0(\vec{n})\text{ch}_1(\vec{n}) - \text{ch}_1(\vec{n})\text{ch}_0(\vec{n}) = 0$.

2.4 The irreducibility of the moduli space

When $\text{MZ}_\mathcal{E}$ intersects the vertical wall L_{w^\pm} , by formula in Lemma 2.5, $\vec{\Upsilon}$ is proportional (up to a positive scalar) to the character on the vertical wall. As $\vec{\rho}$ is to the left of the vertical wall, $\vec{\Upsilon}$ can be written as $a\vec{\rho} - b\vec{\rho}_-$ for some positive numbers a and b . Therefore, $\vec{m} \cdot \vec{\Upsilon} = -b\vec{m} \cdot \vec{\rho}_- > 0$.

When $\text{MZ}_\mathcal{E}$ is to the left of the vertical wall L_{w^\pm} , we have $\frac{\text{ch}_1(E_i)}{\text{ch}_0(E_i)} \leq \frac{\text{ch}_1(\vec{n})}{\text{ch}_0(\vec{n})}$ for $i = 1, 2, 3$. As $\text{ch}_0(\vec{n}) \leq 0$, we have $\begin{vmatrix} \text{ch}_0(\vec{n}) & \text{ch}_1(\vec{n}) \\ \text{ch}_0(E_i) & \text{ch}_1(E_i) \end{vmatrix} > 0$. Since the third term of $\vec{\rho}$ is negative and the character space of \vec{n} is spanned by $\vec{\rho}$ and $(0, n_3, -n_2)$, the character $\vec{\Upsilon}$ can be written as $a\vec{\rho} - b(0, n_3, -n_2)$ for some positive number a and b . As $\vec{\rho}_-$ is in the same chamber as $\vec{\rho} + \epsilon(0, n_3, -n_2)$ and $\vec{m} \cdot \vec{\rho}_- < 0$, we get $\vec{m} \cdot \vec{\Upsilon} > 0$. \square

Now we can give an estimate of the dimension of new stable objects after wall crossing.

Proposition 2.18. *The dimension of the space $\mathbf{Rep}(Q_\mathcal{E}, \vec{n}_w, \alpha_\mathcal{E})^{\vec{\rho}_- - s} \setminus \mathbf{Rep}(Q_\mathcal{E}, \vec{n}_w, \alpha_\mathcal{E})^{\vec{\rho}_+ - s}$ is less than $-\chi(w, w) + \dim G_{\vec{n}_w} - 2$.*

Proof. By Lemma 2.16, the space $\mathbf{Rep}(Q_\mathcal{E}, \vec{n}_w, \alpha_\mathcal{E})^{\vec{\rho}_- - s} \setminus \mathbf{Rep}(Q_\mathcal{E}, \vec{n}_w, \alpha_\mathcal{E})^{\vec{\rho}_+ - s}$ can be covered by the following pieces:

$$\mathbf{Rep}^{\vec{\rho}_- - s} \setminus \mathbf{Rep}^{\vec{\rho}_+ - s} = \bigcup_{\vec{m}} \mathbf{Rep}(Q_\mathcal{E}, \vec{m}, (\vec{n}_w - \vec{m}))^{\vec{\rho}_- - s},$$

where \vec{m} satisfies:

- $\vec{\rho}_- \cdot \vec{m} < 0 = \vec{\rho} \cdot \vec{m}$;
- $\chi(\vec{n}_w - \vec{m}, \vec{m}) \leq 0$.

The second condition is due to Lemma 2.9 and Lemma 2.16. Now similar to the proof of Proposition 2.15, we have

$$\begin{aligned} & \dim \mathbf{Rep}(Q_\mathcal{E}, \vec{m}, \vec{n}_w - \vec{m})_{c,d}^{\vec{\rho}_- - s} \\ & \leq -\chi(\vec{n}_w - \vec{m}, \vec{m}) + \dim G_{\vec{n}_w} - c - d - (\dim G_{(\vec{n}_w - \vec{m})} - c) - (\dim G_{\vec{m}} - d) \\ & \quad + \dim \mathbf{Rep}(Q_\mathcal{E}, \vec{n}_w - \vec{m}, \alpha_\mathcal{E})_c^{\vec{\rho}_- - ss} + \dim \mathbf{Rep}(Q_\mathcal{E}, \vec{m})_d^{\vec{\rho}_- - ss} \\ & \leq -\chi(\vec{n}_w - \vec{m}, \vec{m}) + \dim G_{\vec{n}_w} - \chi(\vec{n}_w - \vec{m}, \vec{n}_w - \vec{m}) - \chi(\vec{m}, \vec{m}) \\ & = -\chi(\vec{n}_w, \vec{n}_w) + \dim G_{\vec{n}_w} + \chi(\vec{m}, \vec{n}_w - \vec{m}) \end{aligned}$$

By Lemma 2.17,

$$\begin{aligned} & -\chi(\vec{m}, \vec{n}_w - \vec{m}) \\ & \geq \chi(\vec{n}_w - \vec{m}, \vec{m}) - \chi(\vec{m}, \vec{n}_w - \vec{m}) \\ & = \chi(\vec{n}_w, \vec{m}) - \chi(\vec{m}, \vec{n}_w) \\ & = 3(\text{ch}_0(\vec{n}_w)\text{ch}_1(\vec{m}) - \text{ch}_0(\vec{m})\text{ch}_1(\vec{n}_w)) \\ & \geq 3 \end{aligned}$$

The last inequality is due to the second statement of Lemma 2.17. \square

2.4 The irreducibility of the moduli space

Now we prove the irreducibility of the moduli space of stable objects. This is well known to hold for moduli of Gieseker stable sheaves. The moduli spaces are given by moduli of quiver representations, so the dimension of each component has a lower bound. The point is, by the previous results, the dimension of new stable objects is smaller than this lower bound, so irreducible component cannot be produced after wall crossing.

Theorem 2.19. *Let w be a primitive character in $K(\mathbf{P}^2)$ such that $\text{ch}_0(w) > 0$. For a generic geometric stability condition $\sigma = \sigma_{s,q}$ with $s < \frac{\text{ch}_1}{\text{ch}_0}(w)$ not on any actual wall of w , the moduli space $\mathfrak{M}_\sigma^{ss}(w)$ is irreducible and smooth.*

Proof. The smoothness is proved in Corollary 2.10. We only need to show the irreducibility.

For any σ , the line $L_{w\sigma}$ intersects some $\text{MZ}_\mathcal{E}$. In fact, we may always choose \mathcal{E} to be $\{O(k-1), O(k), O(k+1)\}$. By Proposition 2.7, $\mathfrak{M}_\sigma^{ss}(w)$ can always be constructed as

$$\mathbf{Rep}(Q_\mathcal{E}, \vec{n}_w, \alpha_\mathcal{E}) //_{\det^{\vec{\rho}_{L_{w\sigma}\mathcal{E}}}} (G_{\vec{n}_w} / \mathbb{C}^\times).$$

In the chamber near the vertical wall, the component that contains $\mathbf{Rep}(Q_\mathcal{E}, \vec{n}_w, \alpha_\mathcal{E})^{\vec{\rho}-s}$ is irreducible since the quotient space $\mathbf{Rep}(Q_\mathcal{E}, \vec{n}_w, \alpha_\mathcal{E})^{\vec{\rho}-s} / G$ is $\mathfrak{M}_{\text{GM}}^s(w)$, which is smooth and connected.

By Proposition 2.18, while crossing an actual wall, the new stable locus $\mathbf{Rep}(Q_\mathcal{E}, \vec{n}_w, \alpha_\mathcal{E})^{\vec{\rho}-s} \setminus \mathbf{Rep}(Q_\mathcal{E}, \vec{n}_w, \alpha_\mathcal{E})^{\vec{\rho}+s}$ in $\mathbf{Rep}(Q_\mathcal{E}, \vec{n}_w, \alpha_\mathcal{E})^{\vec{\rho}-s}$ has codimension greater than 2. On the other hand, since the $\mathbf{Rep}(Q_\mathcal{E}, \vec{n}_w, \alpha_\mathcal{E})$ is a subspace in $\mathbf{Rep}(Q_\mathcal{E}, \vec{n}_w)$ determined by $n_1 n_2 \text{hom}(E_1, E_3)$ equations, each irreducible component has dimension at least

$$n_1 n_2 \text{hom}(E_1, E_2) + n_2 n_3 \text{hom}(E_2, E_3) - n_1 n_2 \text{hom}(E_1, E_3),$$

which is the same as the dimension of $\mathbf{Rep}(Q_\mathcal{E}, \vec{n}_w, \alpha_\mathcal{E})^{\vec{\rho}+s}$ and is greater than the dimension of the new stable locus $\mathbf{Rep}(Q_\mathcal{E}, \vec{n}_w, \alpha_\mathcal{E})^{\vec{\rho}-s} \setminus \mathbf{Rep}(Q_\mathcal{E}, \vec{n}_w, \alpha_\mathcal{E})^{\vec{\rho}+s}$. Since the stable locus is open in $\mathbf{Rep}(Q_\mathcal{E}, \vec{n}_w, \alpha_\mathcal{E})$, the new stable locus is contained in the same irreducible component of $\mathbf{Rep}(Q_\mathcal{E}, \vec{n}_w, \alpha_\mathcal{E})^{\vec{\rho}+s}$. $\mathbf{Rep}(Q_\mathcal{E}, \vec{n}_w, \alpha_\mathcal{E})^{\vec{\rho}-s}$ is still irreducible. Hence the moduli space of Bridgeland stable objects, given as the GIT quotient, is also irreducible. \square

Remark 2.20. *There is natural isomorphism*

$$\mathfrak{M}_{\sigma_{s,q}}^{ss}(w) \simeq \mathfrak{M}_{\sigma_{-s,q}}^{ss}(-\text{ch}_0(w), \text{ch}_1(w), -\text{ch}_2(w))$$

induced by the map $\iota : F \mapsto \mathcal{R}\text{Hom}(F, O)[1]$. In terms of the quiver representation, an object $\mathbf{K}_F \in \mathbf{Rep}(Q_\mathcal{E}, \vec{n}, \alpha_\mathcal{E})^{\vec{\rho}-ss}$:

$$\mathbf{K}_F : E_1 \otimes H_{n_1} \xrightarrow{I_F} E_2 \otimes H_{n_2} \xrightarrow{J_F} E_3 \otimes H_{n_3}$$

is mapped to $\mathbf{K}_{\iota(F)} \in \mathbf{Rep}(Q_{\mathcal{E}^\vee}, (n_3, n_2, n_1), \alpha_{\mathcal{E}^\vee})^{(-\rho_3, -\rho_2, -\rho_1)-ss}$:

$$\mathbf{K}_{\iota(F)} : E_3^\vee \otimes H_{n_3}^* \xrightarrow{J_F^T} E_2^\vee \otimes H_{n_2}^* \xrightarrow{I_F^T} E_1^\vee \otimes H_{n_1}^*.$$

The statement in Theorem 2.19 holds for $\mathfrak{M}_\sigma^s(-w)$ when $\sigma = \sigma_{s,q}$ with $s > \frac{\text{ch}_1}{\text{ch}_0}(w)$.

2.5 Properties of GIT

Birational geometry via GIT has been studied in [DH] by Dolgachev and Hu, [Th] by Thaddeus. Since the theorems in [DH, Th] are stated based on a slightly different set-up, in this section, we recollect some properties from these papers in the language of affine GIT.

Let X be an affine algebraic G -variety, where G is a reductive group and acts on X via a linear representation. Given a character $\rho: G \rightarrow \mathbb{C}^\times$, the (semi)stable locus is written as $X^{\vec{\rho}-s}$ ($X^{\vec{\rho}-ss}$). We write $\mathbb{C}[X]^{G, \chi}$ for the χ -semi-invariant functions on X , in other words, one has

$$f(g^{-1}(x)) = \chi(g) \cdot f(x), \text{ for } \forall g \in G, x \in X.$$

Denote the GIT quotient by $X//_{\vec{\rho}}G := \mathbf{Proj} \bigoplus_{n \geq 0} \mathbb{C}[X]^{G, \vec{\rho}^n}$ and the map from $X^{\vec{\rho}-ss}$ to $X//_{\vec{\rho}}G$ by $F_{\vec{\rho}}$.

In additions, we need the following assumptions on X and G :

1. There are only finite many walls in the space of characters on which there are strictly semistable points, in the chamber we have $X^{\vec{\rho}-s} = X^{\vec{\rho}-ss}$.
2. $X^{\vec{\rho}-s}$ is smooth and the action of G on $X^{\vec{\rho}-s}$ is free.
3. $X//_{\vec{\rho}}G$ is projective and irreducible.
4. The closure of any $X^{\vec{\rho}-s}$ (if non-empty) for any $\vec{\rho}$ is a same irreducible component.
5. Given any point $x \in X$, the set of characters $\{\rho \mid x \in X^{\rho-ss}\}$ is closed.

Let $\vec{\rho}$ be a generic character (i.e. not on any walls) such that $X^{\vec{\rho}-s}$ is non-empty. By assumptions 2 and 3, we have a G -principal bundle $X^{\vec{\rho}-s} \rightarrow X//_{\vec{\rho}}G = X^{\vec{\rho}-s}/G$.

Definition 2.21. Let $\vec{\rho}_0$ be a character of G , we denote $\mathcal{L}_{\vec{\rho}, \vec{\rho}_0}$ to be the line bundle over $X//_{\vec{\rho}}G$ by composing the transition functions of the G -principal bundles with $\vec{\rho}_0$.

In other words, viewing $X^{\vec{\rho}-s}/G$ as a complex manifold, it has an open cover with trivialization of G -fibers. The line bundle $\mathcal{L}_{\vec{\rho}, \vec{\rho}_0}$ is by composing each transition function on the overlap of charts by $\vec{\rho}_0$.

Now we are ready to list some properties from the variation geometric invariant theory.

Proposition 2.22. Let X be an affine algebraic G -variety that satisfies the assumptions 1 to 5, and $\vec{\rho}$ be a generic character. The following properties hold:

1. $\Gamma(X//_{\vec{\rho}}G, \mathcal{L}_{\vec{\rho}, \vec{\rho}_1}^{\otimes n}) \simeq \mathbb{C}[X^{\vec{\rho}-s}]^{G, \vec{\rho}_1^n}$.
2. Let $\vec{\rho}_+$ be a character of G in the same chamber of $\vec{\rho}$, then $\mathbb{C}[X^{\vec{\rho}-s}]^{G, \vec{\rho}_+^n} = \mathbb{C}[X]^{G, \vec{\rho}_+^n}$ for $n \gg 1$ and $\mathcal{L}_{\vec{\rho}, \vec{\rho}_+}$ is ample. Let $\vec{\rho}_0$ be a generic character on the wall of the $\vec{\rho}$ -chamber, then $\mathcal{L}_{\vec{\rho}, \vec{\rho}_0}$ is nef and semi-ample.
3. There is an inclusion $X^{\vec{\rho}_+-ss} \subset X^{\vec{\rho}_0-ss}$ inducing a canonical projective morphism $\text{pr}_+: X//_{\vec{\rho}_+}G \rightarrow X//_{\vec{\rho}_0}G$.

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4. A curve C (projective, smooth, connected) in $X//_{\vec{\rho}_+} G$ is contracted by pr_+ if and only if it is contracted by $X//_{\vec{\rho}_+} G \rightarrow \mathbf{Proj} \bigoplus_{n \geq 0} \Gamma(X//_{\vec{\rho}_+} G, \mathcal{L}_{\vec{\rho}_+, \vec{\rho}_0}^{\otimes n})$.
5. Let $\vec{\rho}_+$ and $\vec{\rho}_-$ be in two chambers on different sides of the wall. Assume that $X^{\vec{\rho}_+ - s}$ and $X^{\vec{\rho}_- - s}$ are both non-empty, then the morphisms $X//_{\vec{\rho}_\pm} G \rightarrow X//_{\vec{\rho}_0} G$ are proper and birational. If they are both small, then the rational map $X//_{\vec{\rho}_-} G \dashrightarrow X//_{\vec{\rho}_+} G$ is a flip with respect to $\mathcal{L}_{\vec{\rho}_+, \vec{\rho}_0}$.

Proof. 1. This is true for a general G -principal bundle by flat descent theorem, see [De] Exposé I, Théorème 4.5.

2 and 3. By the assumption 5, $X^{\vec{\rho} - s} \subset X^{\vec{\rho}_* - ss}$ for $*$ = 0 or +. By the assumption 4, the natural map: $\mathbb{C}[X]^{G, \vec{\rho}_*^n} \rightarrow \mathbb{C}[X^{\vec{\rho} - s}]^{G, \vec{\rho}_*^n} \simeq \Gamma(X//_{\vec{\rho}} G, \mathcal{L}_{\vec{\rho}, \vec{\rho}_*}^{\otimes n})$ is injective for $n \in \mathbb{Z}_{\geq 0}$. Hence the base locus of $\mathcal{L}_{\vec{\rho}, \vec{\rho}_*}$ is empty. $\mathcal{R}(X//_{\vec{\rho}} G, \mathcal{L}_{\vec{\rho}, \vec{\rho}_*}) \simeq \bigoplus_{n \geq 0} \mathbb{C}[X^{\vec{\rho} - s}]^{G, \vec{\rho}_*^n}$ is finitely generated over \mathbb{C} . The canonical morphism $X//_{\vec{\rho}} G \rightarrow \mathbf{Proj} \bigoplus_{n \geq 0} \mathbb{C}[X^{\vec{\rho} - s}]^{G, \vec{\rho}_*^n}$ is birational and projective when $X^{\vec{\rho}_* - s}$ is non-empty. Now we have series of morphisms:

$$\text{pr}_+: X//_{\vec{\rho}} G \rightarrow \mathbf{Proj} \bigoplus_{n \geq 0} \mathbb{C}[X^{\vec{\rho} - s}]^{G, \vec{\rho}_*^n} \rightarrow \mathbf{Proj} \bigoplus_{n \geq 0} \mathbb{C}[X]^{G, \vec{\rho}_*^n} = X//_{\vec{\rho}_*} G.$$

The morphism pr_+ maps each $\vec{\rho}_*$ S-equivariant class to itself set-theoretically. When $\vec{\rho}_+$ is in the same chamber of $\vec{\rho}$, by the assumption 2, this is an isomorphism, implying that $\mathcal{L}_{\vec{\rho}, \vec{\rho}_+}$ must be ample and $\mathbb{C}[X^{\vec{\rho} - s}]^{G, \vec{\rho}_*^n} = \mathbb{C}[X]^{G, \vec{\rho}_*^n}$ for n large enough. By the definition of $\mathcal{L}_{\vec{\rho}, \vec{\rho}_+}$, it extends linearly to a map from the space of \mathbb{R} -characters of G to $\text{NS}_{\mathbb{R}}(X//_{\vec{\rho}} G)$. Since all elements in the $\vec{\rho}$ chamber are mapped into the ample cone, $\vec{\rho}_0$ must be nef.

4. ‘ \Leftarrow ’: The morphism

$$X//_{\vec{\rho}_+} G \rightarrow X//_{\vec{\rho}_0} G = \mathbf{Proj} \bigoplus_{n \geq 0} \mathbb{C}[X]^{G, \vec{\rho}_0^n}$$

factors via the morphism $\mathbf{Proj} \bigoplus_{n \geq 0} \mathbb{C}[X^{\vec{\rho}_+ - s}]^{G, \vec{\rho}_0^n} \rightarrow \mathbf{Proj} \bigoplus_{n \geq 0} \mathbb{C}[X]^{G, \vec{\rho}_0^n}$. If C is contracted at $\mathbf{Proj} \bigoplus_{n \geq 0} \mathbb{C}[X^{\vec{\rho}_+ - s}]^{G, \vec{\rho}_0^n}$, then it is also contracted at $\mathbf{Proj} \bigoplus_{n \geq 0} \mathbb{C}[X]^{G, \vec{\rho}_0^n}$.

‘ \Rightarrow ’: Suppose C is contracted to a point by pr_+ . Let G' be the kernel of $\vec{\rho}_0$, we show that there is a subvariety P in $X^{\vec{\rho}_+ - s}$ such that

I. P is a G' -principal bundle, and the base space is projective, connected;

II. $F_{\vec{\rho}_+}(P) = C$.

Suppose we find such P , then any function f in $\mathbb{C}[X^{\vec{\rho}_+ - s}]^{G, \vec{\rho}_0^n}$ is constant on each G' fiber. Since the base space is projective and connected, it must be a constant on P . Since $F_{\vec{\rho}_+}(P) = C$, the value of f on $F_{\vec{\rho}_+}^{-1}(C)$ is determined by this constant. Hence the canonical morphism contracts C to a point.

We may assume $G' \neq G$, choose N large enough and finitely many f_i 's in $\mathbb{C}[X]^{G, \vec{\rho}_0^N}$ such that $\bigcap_i (V(f_i) \cap F_{\vec{\rho}_0}^{-1}(\text{pr}_+(C)))$ is empty. Since all points in $F_{\vec{\rho}_0}^{-1}(\text{pr}_+(C))$ are S-equivariant in $X^{\vec{\rho}_0 - ss}$, for each point x in $F_{\vec{\rho}_+}^{-1}(C)$, \overline{Gx} contains all minimum orbits Gy in $F_{\vec{\rho}_0}^{-1}(\text{pr}_+(C))$. Choose y in $F_{\vec{\rho}_0}^{-1}(\text{pr}_+(C))$ such that Gy is closed in $X^{\vec{\rho}_0 - ss}$, let P_y be

$$\bigcap_i \{x \in F_{\vec{\rho}_+}^{-1}(C) | f_i(x) = f_i(y)\}.$$

2.6 Walls-crossing as minimal model program

For any $p \in C$, since G is reductive and the G -orbit $\overline{F_{\vec{\rho}_+}^{-1}(p)}$ contains y , there is a subgroup $\beta: \mathbb{C}^\times \rightarrow G$ and $x_p \in F_{\vec{\rho}_+}^{-1}(p)$ such that $y \in \overline{\beta(\mathbb{C}^\times) \cdot \{x_p\}}$. Since $y \in X^{\vec{\rho}_0 - ss}$, there is a $\vec{\rho}_0^N$ -semi-invariant f_i such that $f_i(y) \neq 0$. Therefore $\vec{\rho}_0 \circ \beta \neq 0$, and for any $\vec{\rho}_0$ -semi-invariant function f , $f(x_p) = f(y)$. The point x_p is in P_y and therefore $F_{\vec{\rho}}(P_y) = C$.

Let G'' be the kernel of $\vec{\rho}_0^N$. By the choices of f_i 's, another point x_q on Gx_p is in P_y if and only if they are on the same G'' -orbit. Since G acts freely on all stable points, P_y becomes a G'' principal bundle over base C . As $[G'' : G']$ is finite, we may choose a connected component of P_y such that viewing as a G' -principal bundle, the induced morphism from the base space to C is finite. This component of P_y then satisfies both condition I and II at the beginning.

5. This is due to Theorem 3.3 in [Th]. \square

Remark 2.23. When the difference between $X^{\vec{\rho}_+ - s}$ and $X^{\vec{\rho}_- - s}$ is of codimension two in $X^{\vec{\rho}_+ - s} \cup X^{\vec{\rho}_- - s}$, since $X^{\vec{\rho}_+ - s} \cup X^{\vec{\rho}_- - s}$ is smooth, irreducible and quasi-affine by the second assumption, we have:

$$\mathbb{C}[X^{\vec{\rho}_+ - s}]^{G, \vec{\rho}_+^n} = \mathbb{C}[X^{\vec{\rho}_+ - s} \cup X^{\vec{\rho}_- - s}]^{G, \vec{\rho}_+^n} = \mathbb{C}[X^{\vec{\rho}_- - s}]^{G, \vec{\rho}_+^n} = \mathbb{C}[X]^{G, \vec{\rho}_+^n} \text{ for } n \gg 0.$$

In this case, the birational morphism between $X^{s, \vec{\rho}_+}$ and $X^{s, \vec{\rho}_-}$ identifies $NS_{\mathbb{R}}(X//_{\vec{\rho}_+} G)$ and $NS_{\mathbb{R}}(X//_{\vec{\rho}_-} G)$. It maps $[\mathcal{L}_{\vec{\rho}_+, \vec{\rho}_*}]$ to $[\mathcal{L}_{\vec{\rho}_-, \vec{\rho}_*}]$ for all $\vec{\rho}_*$ in either $\vec{\rho}_+$ and $\vec{\rho}_-$ chamber.

2.6 Walls-crossing as minimal model program

Let w be a primitive character in $K(\mathbf{P}^2)$ such that $\text{ch}_0(w) > 0$. We can run the minimal model program for $\mathfrak{M}_{GM}^s(w)$ via wall crossing on the space of stability conditions.

Theorem 2.24. Adopt the notations as above, the actual walls $L_{w\sigma}$ (chambers) to the left of the vertical wall $L_{w\pm}$ in the $\{1, \frac{\text{ch}_1}{\text{ch}_0}, \frac{\text{ch}_2}{\text{ch}_0}\}$ -plane is one-to-one corresponding to the stable base locus decomposition walls (chambers) on one side (primitive side) of the divisor cone of $\mathfrak{M}_{GM}^s(w)$.

Proof. Suppose $L = L_{w\sigma}$ passes through $\text{MZ}_{\mathcal{E}}$ for an exceptional triple \mathcal{E} . By Lemma 2.5, L associates a character (up to a positive scalar) $\vec{\rho}_L$ to the group $G_{\vec{n}_w}/\mathbb{C}^\times$. By Proposition 2.7, the moduli space $\mathfrak{M}_{\sigma}^{ss}(w)$ is constructed as the quotient space $\mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}_w, \alpha_{\mathcal{E}}) //_{\det^{\vec{\rho}_L}} (G_{\vec{n}_w}/\mathbb{C}^\times)$. We first check that the G -variety $\mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}_w, \alpha_{\mathcal{E}})$ satisfies the assumptions of Proposition 2.22. Assumption 1 is due to Proposition 2.8. Assumption 2 is due to Corollary 2.10 and Remark 2.11. Assumption 3 and 4 are due to Theorem 2.19 and its proof. Assumption 5 is automatically satisfied in our case.

By Definition 2.21, the character $\vec{\rho}_L$ induces a divisor (up to a positive scalar) $[\mathcal{L}_{\vec{\rho}_L, \vec{\rho}_L}]$ on $\mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}_w, \alpha_{\mathcal{E}}) //_{\det^{\vec{\rho}_L}} (G_{\vec{n}_w}/\mathbb{C}^\times)$. We start from the chamber on the left of the vertical wall, where $\mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}_w, \alpha_{\mathcal{E}}) //_{\det^{\vec{\rho}_L}} (G_{\vec{n}_w}/\mathbb{C}^\times)$ is isomorphic to $\mathfrak{M}_{GM}^s(w)$, and vary the stability to the wall near the tangent line of $\bar{\Delta}_0$ across w . At an actual destabilizing wall L , let pr_+ be the morphism

$$\mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}_w, \alpha_{\mathcal{E}}) //_{\det^{\vec{\rho}_{L+}}} (G_{\vec{n}_w}/\mathbb{C}^\times) \rightarrow \mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}_w, \alpha_{\mathcal{E}}) //_{\det^{\vec{\rho}_L}} (G_{\vec{n}_w}/\mathbb{C}^\times)$$

as that in Proposition 2.22. One of three different cases may happen:

1. pr_+ is a small contraction;
2. pr_+ is birational and has an exceptional divisor;
3. all objects in $\mathfrak{M}_L^s(w)$ becomes strictly semistable.

By Proposition 2.18, in Case 1, we get small contractions on both sides. By Property 5 in Proposition 2.22, this is the flip with respect to the divisor $[\mathcal{L}_{\vec{\rho}_{L+}, \vec{\rho}_L}]$. Since the different locus between $\mathbf{Rep}(Q_\mathcal{E}, \vec{n}_w, \alpha_\mathcal{E})^{\vec{\rho}_{L+}}$ and $\mathbf{Rep}(Q_\mathcal{E}, \vec{n}_w, \alpha_\mathcal{E})^{\vec{\rho}_{L-}}$ is of codimension at least 2, their divisor cones are identified with each other as explained in Remark 2.23. In particular, before encountering any wall of Case 2 or 3, the divisor $[\mathcal{L}_{\vec{\rho}_{L+}, \vec{\rho}_L}]$ is identified to a divisor $[\mathcal{L}_{\vec{\rho}_L}]$ on $\mathfrak{M}_{GM}^s(w)$. The flip $\mathfrak{M}_{GM}^s(w) \dashrightarrow \mathfrak{M}_{L-}^s(w)$ is with respect to this divisor.

In Case 2, by Proposition 2.18, the morphism pr_- on the left side

$$\mathbf{Rep}(Q_\mathcal{E}, \vec{n}_w, \alpha_\mathcal{E}) //_{\det^{\vec{\rho}_{L-}}} (G_{\vec{n}_w} / \mathbb{C}^\times) \rightarrow \mathbf{Rep}(Q_\mathcal{E}, \vec{n}_w, \alpha_\mathcal{E}) //_{\det^{\vec{\rho}_L}} (G_{\vec{n}_w} / \mathbb{C}^\times)$$

does not contract any divisors. Hence the Picard number of $\mathbf{Rep}(Q_\mathcal{E}, \vec{n}_w, \alpha_\mathcal{E}) /_{\det^{\vec{\rho}_{L-}}} (G_{\vec{n}_w} / \mathbb{C}^\times)$ is 1. By Property 4 in Proposition 2.22, Case 2 only happens when the canonical model associated to $\mathcal{L}_{\vec{\rho}_L}$ contracts a divisor, in other words, the divisor of $\mathcal{L}_{\vec{\rho}_L}$ on $\mathfrak{M}_{GM}^s(w)$ is on the boundary of the movable cone. The next destabilizing wall on the left corresponds to the zero divisor, it must be Case 3. On the other hand, by Corollary 1.32, Case 3 must happen at a wall before reaching the tangent line. This terminates the whole minimal model program.

In general, if the boundary of the Movable cone is not the same as that of the Nef cone, then Case 2 happens. Otherwise, Case 2 does not happen and the procedure ends up with a Mori fibration of Case 3. \square

Remark 2.25. *On the vertical wall, the morphism pr_+ is the Donaldson-Uhlenbeck morphism. If it contracts a divisor, the vertical wall corresponds to the movable boundary and the minimal model program stops. If pr_+ is a small contraction, the wall-crossing behavior on the other side of the nef cone is the same as the wall-crossing behavior of $\mathfrak{M}_{GM}^s(\text{ch}_0(w), -\text{ch}_1(w), \text{ch}_2(w))$ on the primitive side.*

3 The last wall and criteria for actual walls

In this section, we give a description of the last wall (Section 3.2) and a numerical criteria of actual walls (Section 3.3). Section 3.1 consists of several useful lemmas.

3.1 Stable objects by extensions

The following lemma is useful to construct new stable objects after wall-crossing.

Lemma 3.1. *Let G and F be two $\sigma_{s,q}$ -stable objects of the same phase, in particular, $\sigma_{s,q}$ is on the line L_{GF} . Suppose we have*

$$\phi_{\sigma_{s,q+}}(G) > \phi_{\sigma_{s,q+}}(F), \text{ and } \text{Hom}(G, F[1]) \neq 0.$$

Let f be a non-zero element in $\text{Hom}(G, F[1])$ and C be the corresponding extension of G by F , then C is $\sigma_{s,q+}$ -stable.

3.1 Stable objects by extensions

Proof. By Corollary 1.21 and Proposition 1.26, we may assume that $\sigma_{s,q}$ is in a quiver region $\text{MZ}_{\mathcal{E}}$ so that C , F and G are in the same heart $\mathcal{A}_{\mathcal{E}}[t]$ for a homological shift $t = 0$ or 1. We write σ for $\sigma_{s,q}$, and σ_+ for $\sigma_{s,q+}$.

We prove the lemma by contradiction, suppose D is a σ_+ -stable sub-complex destabilizing C in $\mathcal{A}_{\mathcal{E}}[t]$. We have the following diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & K & \longrightarrow & D & \longrightarrow & I \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & F & \longrightarrow & C & \longrightarrow & G \longrightarrow 0 \end{array}$$

such that the vertical maps are all injective in $\mathcal{A}_{\mathcal{E}}[t]$. Three different cases may happen.

If $I = 0$, then $\phi_{\sigma_+}(K) = \phi_{\sigma_+}(D) \geq \phi_{\sigma_+}(C) > \phi_{\sigma_+}(F)$. But F is also σ_+ -stable, this leads to a contradiction.

If $K = 0$, then either $\phi_{\sigma}(I) < \phi_{\sigma}(G) = \phi_{\sigma}(C)$ or $I = G$. The second case that $I = G$ is impossible since the extension is non-splitting. In the first case, as the phase function is continuous (by the support property), we have $\phi_{\sigma_+}(I) < \phi_{\sigma_+}(C)$. Therefore the object D does not destabilize C at σ_+ , which is a contradiction.

If both K and I are non-zero, then since F and G are σ -stable, $\phi_{\sigma}(I) \leq \phi_{\sigma}(G) = \phi_{\sigma}(C)$ and $\phi_{\sigma}(K) \leq \phi_{\sigma}(F) = \phi_{\sigma}(C)$. When both equalities hold, we have $I = G$ and $K = F$, and in this case, $D = C$. If at least one of the equalities does not hold, then $\phi_{\sigma}(D) < \phi_{\sigma}(C)$. Again by the continuity of the phase function, we see that $\phi_{\sigma_+}(I) < \phi_{\sigma_+}(C)$ and get the contradiction. \square

In general, we also need the following direct sum version, which can be proved in a similar way.

Corollary 3.2. *Let G and F be two $\sigma_{s,q}$ -stable objects of the same phase. Suppose we have*

$$\phi_{\sigma_{s,q+}}(G) > \phi_{\sigma_{s,q+}}(F), \text{ and } \text{Hom}(G, F[1]) = n > 0.$$

Let f be a rank m map in $\text{Hom}(G, F^{\oplus m}[1])$ and C be the object extended by G and $F^{\oplus m}$ via f , then C is $\sigma_{s,q+}$ -stable.

Now we collect some geometric properties of the Le Potier curve. For an exceptional character e , by the Hirzebruch-Riemann-Roch formula, the equation for $L_{e^+e^r}$, i.e. $\chi(-, e') = 0$, in the $\{1, \frac{\text{ch}_1}{\text{ch}_0}, \frac{\text{ch}_2}{\text{ch}_0}\}$ -plane is

$$\text{ch}_0(e) \frac{\text{ch}_2}{\text{ch}_0} - \left(\text{ch}_1(e) + \frac{3}{2} \text{ch}_0(e) \right) \frac{\text{ch}_1}{\text{ch}_0} + \text{ch}_2(e) + \frac{3}{2} \text{ch}_1(e) + \text{ch}_0(e) = 0.$$

In particular, the slope of $L_{e^+e^r}$ is $\frac{\text{ch}_1}{\text{ch}_0}(e) + \frac{3}{2}$. The line $L_{e^+e^r}$ is parallel to $L_{ee(3)}$ and $L_{e^le(3)^r}$. A similar computation shows that the slope of $L_{e^+e^l}$ is $\frac{\text{ch}_1}{\text{ch}_0}(e) - \frac{3}{2}$.

We first want to prove the following result, which will be used to prove Lemma 3.4.

Lemma 3.3. *1. Let e be an exceptional character, then for any point p on the line segment $l_{e^+e^r}$ (not on the boundary), the line L_{ep} intersects the Le Potier curve C_{LP} at two points. In addition, the $\frac{\text{ch}_1}{\text{ch}_0}$ -length of these two points is greater than 3.*

2. Let u and v be two Chern characters with $\text{ch}_0(u), \text{ch}_0(v) > 0$ on the C_{LP} such that their $\frac{\text{ch}_1}{\text{ch}_0}$ -length is greater than 3, then $\chi(u, v) > 0$, $\chi(v, u) > 0$.

3.1 Stable objects by extensions

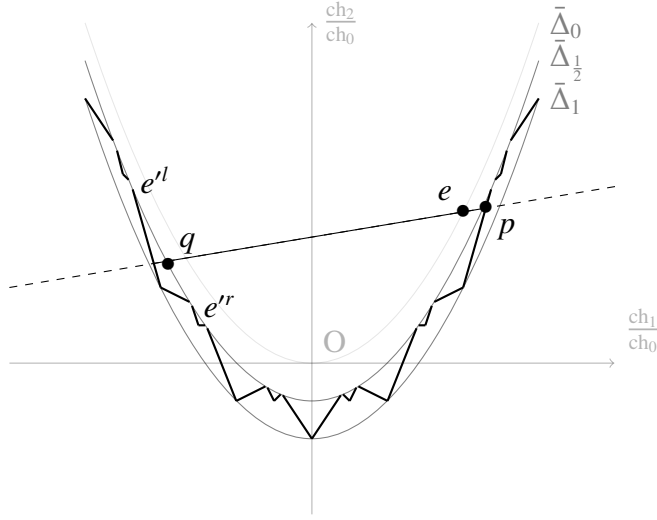


Figure: The intersection of L_{ep} with C_{LP} .

Proof. 1. We first show that L_{ep} only intersects C_{LP} at two points. Since any point on C_{LP} to the right of e^r is above the line L_{ee^r} , we only need to consider points to the left of e .

For any $e^{r'}$ to the left of e that is above L_{ep} , it is also strictly above L_{ee^r} . Since e , e^r and $e(-3)^r$ are collinear, e' is to the left of $e(-3)$. In other words, e' satisfies that $\frac{ch_1}{ch_0}(e') < \frac{ch_1}{ch_0}(e) - 3$. The slope of $L_{ep} >$ the slope of $L_{ee^{l'}}$ $>$ the slope of $L_{e'(3)^r e^{l'}} =$ the slope of $L_{e^{l'} e^{r'}}$. Therefore, L_{ep} does not intersect $l_{e^{l'} e^{r'}}$ or $l_{e^{l'} e^{r'}}$.

For any $e^{l'}$ below L_{ep} to the left of e , the line segments $l_{e^{l'} e^{r'}}$ and $l_{e^{l'} e^{r'}}$ are below $\bar{\Delta}_{\frac{1}{2}}$. The segment of $\bar{\Delta}_{\frac{1}{2}}$ between $e^{l'}$ and $e^{r'}$ is below L_{ep} , hence $l_{e^{l'} e^{r'}}$ and $l_{e^{l'} e^{r'}}$ are below L_{ep} , and they do not intersect L_{ep} . Let q be the intersection point of L_{ep} and $\bar{\Delta}_{\frac{1}{2}}$ (there are two such points and we consider the one to the left of e). When q is not on any segment of $\bar{\Delta}_{\frac{1}{2}}$ between $e^{l'}$ and $e^{r'}$, the intersection points of L_{ep} and C_{LP} are q and p . When q is on the segment between $e^{l'}$ and $e^{r'}$ for an exceptional character e' , the second intersection point is either on $l_{e^{l'} e^{r'}}$ or $l_{e^{l'} e^{r'}}$. So there is only one intersection point other than p .

The points e , e^r and $e(-3)^r$ are collinear, and the $\frac{ch_1}{ch_0}$ -length of e^r and $e(-3)^r$ is 3. Since the $\frac{ch_1}{ch_0}$ -length of $L_{ep} \cap \bar{\Delta}_{\frac{1}{2}}$ is increasing when p is moving from e^r to e^+ , the $\frac{ch_1}{ch_0}$ -length of $L_{ep} \cap \bar{\Delta}_{\frac{1}{2}}$ is greater than 3. Therefore, the $\frac{ch_1}{ch_0}$ -length of $L_{ep} \cap C_{LP}$ is greater than 3.

2. Suppose u is on $\bar{\Delta}_{\frac{1}{2}}$, then the line $\chi(u, -) = 0$ in the $\{1, \frac{ch_1}{ch_0}, \frac{ch_2}{ch_0}\}$ -plane is $L_{uu(-3)}$. Hence the point v in the $\{1, \frac{ch_1}{ch_0}, \frac{ch_2}{ch_0}\}$ -plane is above $L_{uu(-3)}$. As $ch_0(u)$ and $ch_0(v)$ are positive, $\chi(u, v) > 0$. $\chi(v, u) > 0$ can be proved similarly.

Suppose u is on $l_{e^+ e^r}$ for an exceptional e , we first show that $\chi(u, v) > 0$. The line $\chi(u, -) = 0$ passes through e , and both e^r and e^l are below the line $\chi(u, -) = 0$. By the $\frac{ch_1}{ch_0}$ -length assumption, v is above both L_{ee^r} and L_{ee^l} . Therefore, v is above the line $\chi(u, -) = 0$. Since $ch_0(u)$ and $ch_0(v)$ are positive, $\chi(u, v) > 0$.

The line $\chi(-, u) = 0$ in the $\{1, \frac{ch_1}{ch_0}, \frac{ch_2}{ch_0}\}$ -plane passes through $e(3)$, and intersects $l_{e(3)e(3)^r}$. If v is on the line segment $l_{e(3)e(3)^r}$, by the case that u is on $l_{e^+ e^r}$, we get $\chi(v, u) > 0$. If v is not on the line segment $l_{e(3)e(3)^r}$, then by the assumption on the $\frac{ch_1}{ch_0}$ -length, v is above the curve $\chi(-, u) = 0$, we also get $\chi(v, u) > 0$.

The case that u is on $l_{e^+ e^l}$ can be proved in the same way. \square

3.1 Stable objects by extensions

Now we can state an important lemma. A similar definition also appears in [CH2].

Lemma 3.4. *Let u and v be Chern characters such that:*

1. *u and v are not inside the Le Potier cone.*
2. *$\Delta(v, u) \geq 0$.*
3. *In the $\{1, \frac{\text{ch}_1}{\text{ch}_0}, \frac{\text{ch}_2}{\text{ch}_0}\}$ -plane, L_{uv} intersects C_{LP} at two points and the $\frac{\text{ch}_1}{\text{ch}_0}$ -length between them is greater than 3.*

Then we have

$$\chi(u, v), \chi(v, u) < 0.$$

Remark 3.5. *When both $\text{ch}_0(u)$ and $\text{ch}_0(v)$ are 0, the third condition does not make sense. But the statement still holds if the first two conditions hold. To see this, note that by the second condition,*

$$\chi(v, w) = \chi(w, v) = -2\Delta(v, w) = -\text{ch}_1(v)\text{ch}_1(w) \leq 0.$$

Now the first condition implies that $\text{ch}_1(w)$ and $\text{ch}_1(v)$ are both non-zero, so

$$-\text{ch}_1(v)\text{ch}_1(w) < 0.$$

Proof. By the first condition, u and v are below $\bar{\Delta}_1$. By the third condition, let f_1 and f_2 be two characters corresponding to the intersection points of L_{uv} and C_{LP} such that $\text{ch}_0(f_1) > 0$, $\text{ch}_0(f_2) > 0$ and $\frac{\text{ch}_1}{\text{ch}_0}(f_1) > \frac{\text{ch}_1}{\text{ch}_0}(f_2)$.

We may assume that $v = a_1 f_1 - a_2 f_2$ and $u = b_1 f_1 - b_2 f_2$ for some non-zero real numbers a_1, a_2, b_1, b_2 , since u and v are not inside Cone_{LP} , we see a_1, a_2 have the same sign and b_1, b_2 have the same sign. Moreover, by the second condition, we have

$$\Delta(v + au, v + au) \geq \Delta(v - au, v - au) \tag{2}$$

for any positive number a . Hence a_1, a_2, b_1, b_2 all have the same sign. Without loss of generality, we may assume they are all positive.

As f_i is on C_{LP} , we have

$$\chi(f_1, f_1), \chi(f_2, f_2) \leq 0.$$

By the third condition, the $\frac{\text{ch}_1}{\text{ch}_0}$ -distance of f_1 and f_2 is greater than 3. By Lemma 3.3,

$$\chi(f_1, f_2) > 0, \chi(f_2, f_1) > 0.$$

Combining these results, we have

$$\chi(u, v) \leq -b_1 a_2 \chi(f_1, f_2) - b_2 a_1 \chi(f_2, f_1) < 0,$$

and

$$\chi(v, u) \leq -b_2 a_1 \chi(f_1, f_2) - b_1 a_2 \chi(f_2, f_1) < 0.$$

□

3.2 The Last Wall

Note that if we have stable objects A and B of characters u and v respectively, satisfying the conditions in the lemma, then the lemma implies that $\text{Ext}^1(A, B) > 0$ and $\text{Ext}^1(B, A) > 0$. By Lemma 3.1, this implies the existence of stable objects as extensions on both sides. This observation will be used in the proof of the last wall to show the non-emptiness of the moduli, and in the proof of the actual walls to show the existence of objects destabilized on each side of the wall.

3.2 The Last Wall

In this section, we describe the last wall for a given character w that is not inside the Le Potier cone Cone_{LP} . By the *last wall* of w , we mean that for $P \in \bar{\Delta}_{<0}$, there is σ_P -stable objects of character w or $w[1]$ if and only if P is above the last wall. By result from Section 3, this wall corresponds to the boundary of the effective cone when running MMP. The last wall is first computed in [CHW] and [Wo] by Coskun, Huizenga and Woolf. We would like to state the result based on our set-up and give a different proof. To describe the last wall for character w , we first define the exceptional bundle associated to w .

Definition 3.6. Let E be an exceptional bundle, we define \mathfrak{R}_E to be the closure of the region bounded by $L_{e(-3)^r ee^r}$, $l_{e^r e^+}$, $l_{e^+ e^l}$ and $L_{e^l e(-3)e(-3)^l}$ in the $\{1, \frac{\text{ch}_1}{\text{ch}_0}, \frac{\text{ch}_2}{\text{ch}_0}\}$ -plane (see the picture below). Symmetrically, we define \mathfrak{Q}_E to be the closure of the region bounded by $L_{E(3)^l ee^l}$, $l_{e^l e^+}$, $l_{e^+ e^r}$ and $L_{e^r E(3)E(3)^r}$ in the $\{1, \frac{\text{ch}_1}{\text{ch}_0}, \frac{\text{ch}_2}{\text{ch}_0}\}$ -plane.

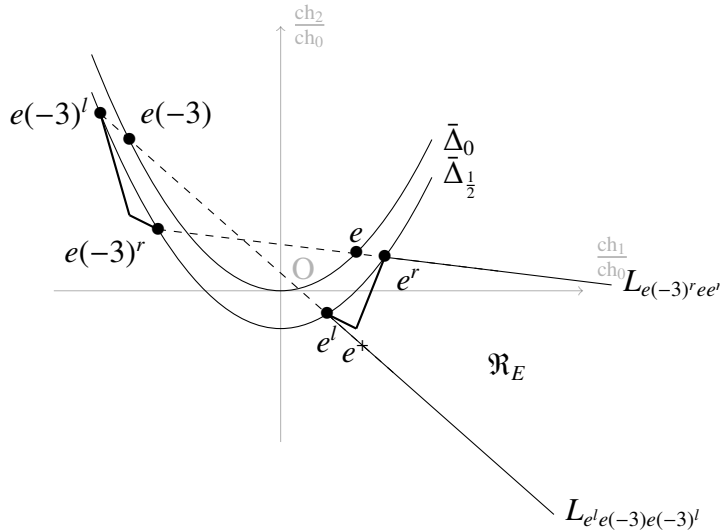


Figure: The region of \mathfrak{R}_E .

The following property translates an important technical result in [CHW] into our set-up.

Proposition 3.7 (Theorem 4.1 in [CHW]). *The regions associated to the exceptional bundles cover all rational points not above the Le Potier curve.*

$$\bigsqcup_{E \text{ exc}} \mathfrak{R}_E \supset \text{P}(\text{K}(\mathbf{P}^2)) \setminus \tilde{\mathcal{C}}_{LP}.$$

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A similar statement holds for \mathfrak{Q}_E .

Proof. Let w be a reduced character in $P(K(\mathbf{P}^2))$ not above C_{LP} . There is a unique line $L_w^{\frac{\bar{\Delta}_1}{2}}$ through w on the $\{1, \frac{\text{ch}_1}{\text{ch}_0}, \frac{\text{ch}_2}{\text{ch}_0}\}$ -plane such that it intersects with $\bar{\Delta}_{\frac{1}{2}}$ at two points f_1 and f_2 , both of which are to the left of w and their $\frac{\text{ch}_1}{\text{ch}_0}$ -length is 3. Let f_1 be the points with larger $\frac{\text{ch}_1}{\text{ch}_0}$. By Theorem 4.1 in [CHW], there is a unique exceptional bundle E such that on the curve $\bar{\Delta}_{\frac{1}{2}}$, f_1 is on the segment between e^l and e^r . For any character u on the line $L_{f_1 f_2}$, we have $\chi(f_1, u) = 0$, hence $\chi(f_1, w) = 0$. The points e^r and e^l are on the different sides of the line $\chi(-, w) = 0$, therefore

$$\chi(e^l, w) \cdot \chi(e^r, w) < 0.$$

Note that the boundary $L_{E(-3)^r E E^r}$ is the line: $\chi(e^r, -) = 0$, and the boundary $L_{E^l E(-3) E(-3)^l}$ is the line: $\chi(e^l, -) = 0$. Hence, w is in \mathfrak{R}_E . \square

Remark 3.8. It is possible to show that $L_w^{\frac{\bar{\Delta}_1}{2}}$ must intersect a line segment $l_{e^l e^r}$ without using Theorem 4.1 in [CHW], but the argument is rather involved. The sketch of argument is as follows: 1. If $L_w^{\frac{\bar{\Delta}_1}{2}}$ does not intersect any line segment $l_{e^l e^r}$, then for any exceptional bundle E with character below $L_w^{\frac{\bar{\Delta}_1}{2}}$, by Proposition 1.27, $\mathfrak{M}_\sigma^s(w)$ is empty for σ below L_{wE} . Therefore, $\mathfrak{M}_\sigma^s(w)$ is empty for σ below $L_w^{\frac{\bar{\Delta}_1}{2}}$. 2. By the same argument for the last wall and Lemma 3.4, $\mathfrak{M}_\sigma^s(w)$ is non-empty for σ on $L_w^{\frac{\bar{\Delta}_1}{2}}$. This leads to the contradiction.

Thanks to this result, we can introduce the following definition, which will be related to the last wall.

Definition 3.9. Let w be a character not inside Cone_{LP} (see Definition 1.6), we define the exceptional bundle E_w associated to w to be the unique one such that \mathfrak{R}_{E_w} contains w . Similarly we have the definition of $E_w^{(rhs)}$ according to \mathfrak{Q}_E .

Remark 3.10 (Torsion Case). In the case that $\text{ch}_0(w) = 0$ and $\text{ch}_1(w) > 0$, E_w is the unique exceptional bundle such that

$$\text{the slope of } L_{e(-3)e^l} < \frac{\text{ch}_2}{\text{ch}_1}(w) < \text{the slope of } L_{ee^r}.$$

The bundle $E_w^{(rhs)}$ is not defined in the torsion case.

Now we can state the location of the last wall.

Definition 3.11. Let w be a character (not necessarily primitive) not inside Cone_{LP} (may be on the boundary but not at the origin, see Definition 1.6) and $E = E_w$ be its associated exceptional vector bundle. We define the last wall L_w^{last} of w according to three different cases:

1. If w is above $L_{e^+e(-3)^+}$, then $L_w^{\text{last}} := L_{we}$.
2. If w is below $L_{e^+e(-3)^+}$, then $L_w^{\text{last}} := L_{we(-3)}$.

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3. If w is on $L_{e^+e(-3)^+}$, then $L_w^{\text{last}} := L_{e^+e(-3)^+}$.

The last wall $L_w^{\text{right-last}}$ on the right side to the vertical wall is defined similarly by using $E^{(rhs)}$. The torsion character does not have $E^{(rhs)}$ or $L^{\text{right-last}}$.

In the cartoon below, F_i is of Case i in the definition respectively.

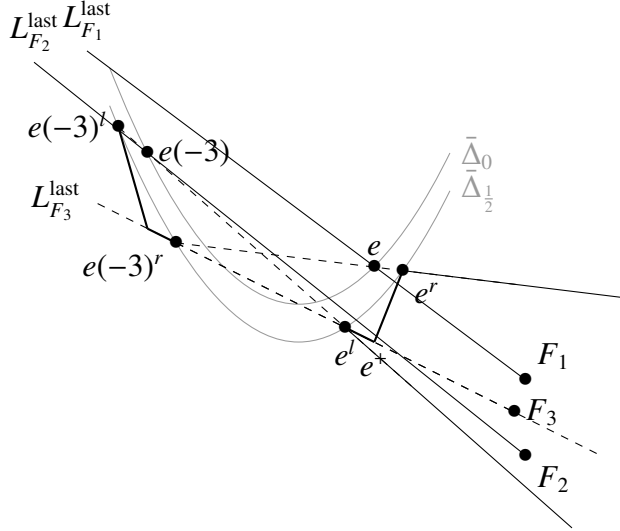


Figure: Three different cases of the last wall.

The following lemma shows that for stability conditions below the wall L_w^{last} ($L_w^{\text{right-last}}$), there is no stable object with character w .

Lemma 3.12. *Let w be a character in $K(\mathbf{P}^2)$ not inside Cone_{LP} ; σ be a geometric stability condition in $\bar{\Delta}_{<0}$ below L_w^{last} or $L_w^{\text{right-last}}$. Then $\mathfrak{M}_\sigma^s(w)$ and $\mathfrak{M}_\sigma^s(-w)$ are both empty.*

Proof. We prove the lemma in the case for L_w^{last} . The $L_w^{\text{right-last}}$ case can be proved similarly. We may assume $\text{ch}_0(w) \geq 0$, since otherwise $\mathfrak{M}_\sigma^s(w)$ is empty when σ is to the left of the vertical wall L_{w^\pm} . When w is of Case 1 or 3 in the Definition 3.11, the statement follows from Proposition 1.27 directly.

When w is of Case 2 in Definition 3.11, we have $\chi(w, E_w(-3)) < 0$. For any σ -stable F with character w , $\text{Hom}(F, E_w(-3)[t])$ may be nonzero only when $0 \leq t \leq 3$. Since F is in $\text{Coh}_{\#s_\sigma}$, $\text{Hom}(E_w, H^{-1}(F)) = 0$. By Serre duality,

$$\text{hom}(F, E_w(-3)[3]) = \text{hom}(E_w, F[-1]) = \text{hom}(E_w, H^{-1}(F)) = 0.$$

On the other hand, when σ is below L_w^{last} and inside $\bar{\Delta}_{<0}$, by Corollary 1.19, $E_w(-3)[1]$ is σ -stable. By Lemma 1.17, $\phi_\sigma(E_w(-3)[1]) < \phi_\sigma(F)$. Therefore, $\text{Hom}(F, E_w(-3)[1]) = 0$. This leads to a contradiction to the inequality that $\chi(w, E_w(-3)) < 0$. \square

The existence of stable objects before the last wall is more complicated. This is first proved by Coskun, Huizenga and Woolf. The authors write down the generic slope stable coherent sheaves build by exceptional bundles and show that these objects do not get destabilized before the last wall. Our approach is more close to the idea of Bayer and

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Macrì for K3 surfaces. We aim to show that for each wall-crossing before the last wall, new stable objects (extended by two objects) are generated on both sides, hence the moduli space is non-empty. We may benefit from this approach since the similar techniques can be applied in the criteria for actual walls.

Lemma 3.13. *Let w be a character $K(\mathbf{P}^2)$ with $\text{ch}_0(w) > 0$ and not inside Cone_{LP} . Let σ be a geometric stability condition. Assume that the wall $L_{w\sigma}$ is between the vertical wall $L_{w\pm}$ and*

- L_w^{last} , when w is of Case 1 or 2 in Definition 3.11;
- $L_{wE_w(-3)}$, when w is of Case 3 in Definition 3.11.

Let $v \in K(\mathbf{P}^2)$ be a character on $L_{w\sigma}$ such that $\text{ch}_0(v) \geq 0$ and $\frac{\text{ch}_1(v)}{\text{ch}_0(v)} > \frac{\text{ch}_1(w)}{\text{ch}_0(w)}$, then the wall $L_{v\sigma}$ is between $L_{v\pm}$ and L_v^{last} .

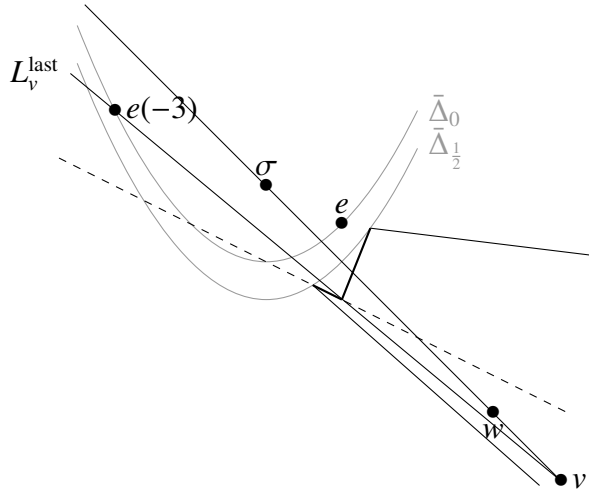


Figure: $L_{v\sigma}$ is between $L_{v\pm}$ and L_v^{last} .

Proof. By the definition of \Re_E and the assumptions on $L_{w\sigma}$, the slope of $L_{w\sigma}$ is less than the slope of $L_{e_w e_w^r}$. As $\frac{\text{ch}_1(v)}{\text{ch}_0(v)} > \frac{\text{ch}_1(w)}{\text{ch}_0(w)}$ and $\text{ch}_0(v) \geq 0$, v is to the right of w in the $\{1, \frac{\text{ch}_1}{\text{ch}_0}, \frac{\text{ch}_2}{\text{ch}_0}\}$ -plane. Therefore, either v is in \Re_{E_w} , or $\frac{\text{ch}_1(E_v)}{\text{ch}_0(E_v)} < \frac{\text{ch}_1(E_w)}{\text{ch}_0(E_w)}$. L_{vE_w} is either L_v^{last} or between L_v^{last} and $L_{v\pm}$.

When w is of Case 1 in the Definition 3.11, E_w is below $L_{vw\sigma}$, therefore $L_{vw\sigma}$ is between the wall L_{vE_w} and $L_{v\pm}$, and the conclusion follows.

When w is of Case 2 and 3 in the Definition 3.11, v is in \Re_{E_w} of Case 3 or E_v has slope less than E_w . In either case, $L_{vE_w(-3)}$ is either L_v^{last} or between L_v^{last} and $L_{v\pm}$. $E_w(-3)$ is below $L_{vw\sigma}$, therefore $L_{vw\sigma}$ is between the wall $L_{vE_w(-3)}$ and $L_{v\pm}$, hence between the wall L_v^{last} and $L_{v\pm}$. \square

Theorem 3.14. *Let w be a character in $K(\mathbf{P}^2)$ not inside the Le Potier cone Cone_{LP} ; σ be a geometric stability condition in $\bar{\Delta}_{<0}$ between L_w^{last} and $L_w^{\text{right-last}}$. When σ is not on the vertical wall $L_{w\pm}$, either $\mathfrak{M}_\sigma^s(w)$ or $\mathfrak{M}_\sigma^s(-w)$ is non-empty.*

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The proof of the theorem is rather involved, so we want to sketch the idea here. The aim is to show the existence of new stable objects given by extensions after wall crossing. First, on any wall before the last wall, the pair of destabilizing Chern characters w' and $w - w'$ are between their own last walls and the vertical walls. By induction on the discriminant, there exists stable objects with characters w' and $w - w'$. Then by Lemma 3.1 and 3.4, we show that these objects have non-trivial extensions, and will extend to stable objects after each wall-crossing. However, several different cases may happen so that the idea cannot work directly. When one of the destabilizing characters is proportional to an exceptional character, Condition 1 in Lemma 3.4 fails and we need other ways to show $\chi(w', w - w') < 0$. The most complicated case is when w' is of higher rank and $L_{w-w'}^{\text{right-last}}$ is $L_{ww'}$ (Case 3.II.2 in the proof). In this case, $\mathfrak{M}_{\sigma}^{ss}((w' - w)[1])$ may not contain any stable objects. To deal with that, we adjust $w' - w$ to another character \tilde{w} on $L_{ww'}$ so that \tilde{w} is of positive rank and $L_{\tilde{w}}^{\text{right-last}}$ is not $L_{ww'}$. The details of the argument are as follows.

Proof. Assume the proposition does not hold, among all the characters w not inside the Le Potier cone, such that $\mathfrak{M}_{\sigma'}^s(w)$ and $\mathfrak{M}_{\sigma'}^s(-w)$ are both empty for some σ' in $\bar{\Delta}_{<0}$ between L_w^{last} and $L_w^{\text{right-last}}$, we may choose w with the minimum discriminant Δ . We may assume that $\text{ch}_0(w) \geq 0$. When σ' is to the left of $L_{w\pm}$, $\mathfrak{M}_{\sigma',q}^s(w)$ contains Gieseker-Mumford stable objects for $q \gg \frac{s^2}{2}$ and $s < \frac{\text{ch}_1(w)}{\text{ch}_0(w)}$. $\mathfrak{M}_{\sigma',q}^s(w)$ is not empty by Theorem 1.8. There is a ‘last wall’ $L_{\sigma w}$ prior to L_w^{last} such that $\mathfrak{M}_{\sigma+}^s(w)$ is non-empty, on the wall all objects in $\mathfrak{M}_{\sigma}^{ss}(w)$ are strictly semistable, and $M_{\sigma-}^{ss}(w)$ is empty. There are three main different cases according to the number of exceptional characters on $L_{\sigma w}$.

Case 1. There is no exceptional character on $L_{\sigma w}$. Let F be a σ_+ -stable object of character w , then F is destabilized by a σ -stable object G with $\tilde{v}(G) = w'$ on the line segment $l_{\sigma w}$. $\mathfrak{M}_{\sigma}^{ss}(w - w')$ is not empty since it contains F/G . Since there is no exceptional character on $L_{\sigma w}$, the wall $L_{\sigma w}$ is not the last wall $L_{w-w'}^{\text{last}}$ or $L_{w-w'}^{\text{right-last}}$ for $w - w'$. By Corollary 1.30, $w - w'$ is not inside Cone_{LP} . By Lemma 3.12, $L_{\sigma w}$ is between $L_{w-w'}^{\text{last}}$ and $L_{w-w'}^{\text{right-last}}$. Corollary 3.10 in [BMS] implies $\Delta(w') < \Delta(w)$. By induction on Δ and the fact that $L_{\sigma(w-w')}$ is not the vertical wall, we can assume that $\mathfrak{M}_{\sigma}^s(w - w')$ is non-empty.

We check that the pair w' and $w - w'$ satisfies the conditions in Lemma 3.4:

1. Note that $\mathfrak{M}_{\sigma}^s(w - w')$ and $\mathfrak{M}_{\sigma}^s(w')$ are non-empty, w' and $w - w'$ are not exceptional. By Lemma 1.30, both w' and $w - w'$ are not inside Cone_{LP} .
2. $w' + a(w - w')$ is outside the cone $\Delta_{\leq 0}$ for any $a \geq 0$. Since $L_{w\sigma}$ intersects $\bar{\Delta}_{<0}$, $w' - a(w - w')$ belongs to $\bar{\Delta}_{<0}$ for some $a > 0$.

$$\Delta(w' - a(w - w')) = \Delta(w') + \Delta(w - w') - 2a\Delta(w', w - w') < 0$$

implies $\Delta(w', w - w') \geq 0$.

3. When w is not right orthogonal to E_w , by Lemma 3.3, the $\frac{\text{ch}_1}{\text{ch}_0}$ -length of $L_{wE_w} \cap C_{LP}$ is greater than 3. Hence, the $\frac{\text{ch}_1}{\text{ch}_0}$ -length of $L_{w\sigma} \cap C_{LP}$ is greater than 3. When w is right orthogonal to E_w , note that w' is not in the triangle area $\text{TR}_{we_w e_w^+}$, since otherwise, $E_{w'} = E_w$ and $L_{w'\sigma}$ is to the left of $L_{w'E_w}$, by Proposition 1.27, $\mathfrak{M}_{\sigma}^s(w')$ is empty, there is no σ -stable object G to destabilize F . Now since the $\frac{\text{ch}_1}{\text{ch}_0}$ -length of $L_{wE_w} \cap C_{LP}$ is greater than 3, the $\frac{\text{ch}_1}{\text{ch}_0}$ -length of $L_{w\sigma} \cap C_{LP}$ is greater than 3.

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Now by Lemma 3.4, we have $\chi(w', w - w') < 0$. For σ -stable objects F' and F'' with characters w' and $w - w'$ respectively, and $i \neq 0, 1, 2$, $\text{Hom}(F', F''[i]) = 0$ since F' and F'' are in a same heart and in addition by Serre duality. These imply $\text{Hom}(F', F''[1]) \neq 0$. Now by Lemma 3.1, the non-trivial extension of F' by F'' is σ_- -stable, therefore $\mathfrak{M}_{\sigma_-}^s(w)$ is non-empty, which contradicts to the assumption that $L_{\sigma w}$ is the last wall.

Case 2: There are more than two exceptional characters on $L_{\sigma w}$. This can only happens when $L_{w\sigma}$ is the line $\chi(E, -) = 0$ for exceptional bundle $E = E_w$. In this case, w is of Case 3 in Definition 3.11, $L_{\sigma w}$ is L_w^{last} .

Case 3: There are one or two exceptional characters on $L_{\sigma w}$.

Similar to Case 1, we consider the character w' . We first prove the ‘lower rank wall’ case, i.e. $\text{ch}_0(w') \leq \text{ch}_0(w)$. In this case, since $\phi_{\sigma+}(w) < \phi_{\sigma+}(w - w')$, the character $w - w'$ satisfies the condition in Lemma 3.13, therefore $\mathfrak{M}_{\sigma}^s(w - w')$ is non-empty by induction on Δ . We only need to show $\chi(w', w - w') < 0$ so that by the same argument of the last paragraph in Case 1, $\mathfrak{M}_{\sigma_-}^s(w)$ is non-empty. If w' is not proportional to any exceptional character, then the proof in Case 1 works, and the pair w' and $w - w'$ still satisfies the conditions in Lemma 3.4. If w' is proportional to an exceptional character E , since $L_{E(w-w')}$ is not $L_{(w-w')}^{\text{last}}$, $\chi(E, w - w') < 0$. Therefore, $\chi(E, w - E) < 0$ and $\mathfrak{M}_{\sigma_-}^s(E, w - E)$ is non-empty. This completes the argument for the lower rank case.

Now we may assume $\text{ch}_0(w') > \text{ch}_0(w)$ and let $w'' = w' - w$, then $\text{ch}_0(w'') > 0$. On the $\{1, \frac{\text{ch}_1}{\text{ch}_0}, \frac{\text{ch}_2}{\text{ch}_0}\}$ -plane, w' and w'' are in different components of $L_{w\sigma} \cap \bar{\Delta}_{\geq 0}$. If $\mathfrak{M}_{\sigma}^s(w''[1])$ is non-empty, then the argument for the lower rank case still works and implies that $\mathfrak{M}_{\sigma_-}^s(w)$ is non-empty. On the other hand, by induction on Δ , Proposition 2.15 and Proposition 2.18, the semistable locus $\mathfrak{M}_{\sigma}^{ss}(w''[1])$ is non-empty. So the only remaining case to consider is that $L_{w\sigma}$ is the right last wall $L_{w''}^{\text{right-last}}$ for w'' .

Case 3.I: w'' is proportional to an exceptional character E : $w'' = a\tilde{v}(E)$. Since E is to the left of E_w , we have $\chi(w, E) > 0$, this implies

$$\chi(w', E) > \chi(w'', E) = a\chi(E, E) = a.$$

By Corollary 1.19, both G and $E[1]$ are σ -stable in a same heart, this implies $\text{Hom}(G, E) = \text{Hom}(G, (E[1])[-1]) = 0$. Therefore,

$$\text{ext}^1(G, E[1]) = \text{hom}(G, E[2]) \geq \chi(w', E) > a.$$

By Corollary 3.2, there exists σ_- -stable object extended by G and $E^{\oplus a}[1]$.

Case 3.II: w'' is not proportional to any exceptional character. As $L_{w''}^{\text{right-last}} = L_{w''\sigma}$, and there are at most two exceptional characters on $L_{w''\sigma}$ by assumption, w'' is not of Case 3 in Definition 3.11, either $E_{w''}^{(\text{rhs})}$ or $E_{w''}^{(\text{rhs})}(3)$ is on the line segment $l_{ww''}$.

Case 3.II.1: w'' is of (right side) Case 2 in Definition 3.11 and $\tilde{v}(E_{w''}^{(\text{rhs})}(3))$ is on $l_{ww''}$. The character w can be written as

$$a\tilde{v}(E_{w''}^{(\text{rhs})}(3)) - bw''$$

for some positive numbers a and b . Since $\chi(E_{w''}^{(\text{rhs})}(3), w'') < 0$, we have

$$\chi(E_{w''}^{(\text{rhs})}(3), w) > 0.$$

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This implies $\frac{\text{ch}_1}{\text{ch}_0}(E_{w''}^{(\text{rhs})}(3)) \leq \frac{\text{ch}_1}{\text{ch}_0}(E_w)$. As $E_{w''}^{(\text{rhs})}(3)$ is above L_w^{last} , it must be E_w . On the other hand, as $\chi(E_w, w) = \chi(E_{w''}^{(\text{rhs})}(3), w) > 0$, w is of Case 1 in Definition 3.11. The wall $L_{wE_{w''}^{(\text{rhs})}(3)w}$ is just the last wall L_w^{last} of w .

Case 3.II.2: w'' is of (right side) Case 1 in Definition 3.11 and $E = \tilde{v}(E_{w''}^{(\text{rhs})})$ is on $l_{ww''}$. By Definition 3.11, $\chi(w'', E) > 0$. Consider the character $\tilde{w} := w'' - \chi(w'', E)\tilde{v}(E)$, we have $\chi(\tilde{w}, E) = 0$, therefore \tilde{w} is on the line $L_{e^r(e(3))^l}$.

The character w must be above $L_{EE(3)}$, otherwise L_{wE} is the last wall L_w^{last} . The intersection of $L_{wE} \cap L_{e^r(e(3))^l}$ is outside the cone $\bar{\Delta}_{<0}$ and on the different side of w in the $\{1, \frac{\text{ch}_1}{\text{ch}_0}, \frac{\text{ch}_2}{\text{ch}_0}\}$ -plane. As w'', E and \tilde{w} are on the same component of $L_{wE} \cap \bar{\Delta}_{\geq 0}$, $\text{ch}_0(\tilde{w})$ is greater than 0. The character $w + \tilde{w} = w'' - \chi(w'', E)\tilde{v}(E)$ is on the line segment $l_{ww''}$.

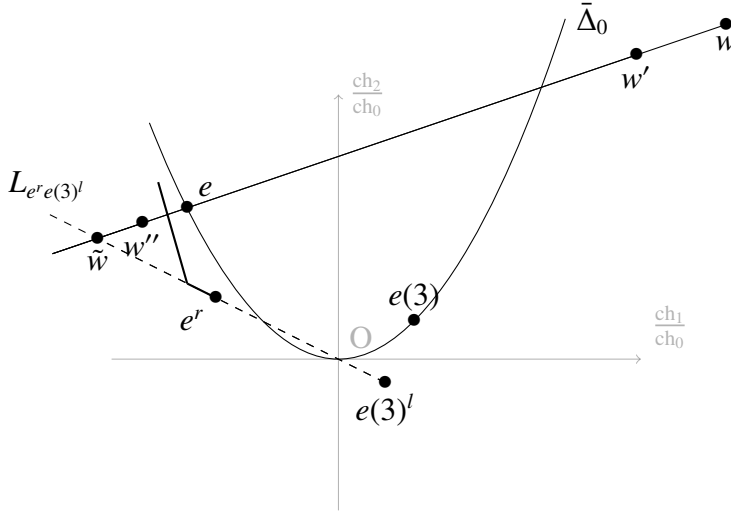


Figure: Definition of \tilde{w} .

When w' is not proportional to any exceptional character, $w + \tilde{w}$ is outside Cone_{LP} and on the same component of $L_{wE} \cap \bar{\Delta}_{\geq 0}$ as w . Since the line segment $l_{\tilde{w}(w+\tilde{w})}$ intersects $\bar{\Delta}_{<0}$, $\Delta(\tilde{w}, w + \tilde{w}) < 0$. This implies $\Delta(\tilde{w}) < \Delta(w)$ and $\Delta(w + \tilde{w}) < \Delta(w)$. As $\mathfrak{M}_{\sigma}^s(w')$ is non-empty, by Lemma 3.13 and induction on Δ , $\mathfrak{M}_{\sigma}^s(w + \tilde{w})$ is non-empty. As $\chi(\tilde{w}, E) = 0$, $L_{\tilde{w}E}$ is not the last wall $L_{\tilde{w}}^{\text{right-last}}$ for \tilde{w} . By induction on Δ , $\mathfrak{M}_{\sigma}^s(-\tilde{w})$ is non-empty. The character pair $w + \tilde{w}$ and $-\tilde{w}$ satisfy the conditions in Lemma 3.4, hence $\chi(w + \tilde{w}, \tilde{w}[1]) < 0$. $\mathfrak{M}_{\sigma-}^s(w)$ is non-empty by Lemma 3.1.

When w' is $\tilde{v}(E')$ for an exceptional bundle E' , as E' is to the right of $E(3)$, we have $\chi(E, E') > 0$. Hence,

$$\chi(w + \tilde{w}, E') = \chi(E' - \chi(w'', E)\tilde{v}(E), E') = 1 - \chi(w'', E)\chi(E, E') \leq 0.$$

This implies the characters $w + \tilde{w}$ and $\tilde{v}(E')$ are on two different sides of $L_{e^r+e^l}$. Therefore, $w + \tilde{w}$ is not inside Cone_{LP} . The rest of the argument is the same as the case when w' is not proportional to exceptional character.

Up to now, we finish the argument for the case that σ is on the left side of $L_{w\pm}$. When σ is on the right side of $L_{w\pm}$, the statement follows from the symmetric property ($\text{ch}_0(w) > 0$):

$$\mathfrak{M}_{\sigma}^s(w) \simeq \mathfrak{M}_{\sigma'}^s(w'[1]), \quad F \mapsto \mathcal{R}\text{Hom}(F, \mathcal{O})[1],$$

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where σ' is with parameter $(-s_\sigma, q_\sigma)$ and $w' = (\text{ch}_0(w), -\text{ch}_1(w), \text{ch}_2(w))$.

3.3 The criteria for actual walls

In this section we give a numerical criteria for actual walls of a given Chern character. In the $\{1, \frac{\text{ch}_1}{\text{ch}_0}, \frac{\text{ch}_2}{\text{ch}_0}\}$ -plane, the actual wall for w is the potential wall $L_{w\sigma}$ where new stable objects are produced on both sides and curves are contracted on at least one side. When σ is to the left of the vertical wall $L_{w\pm}$, one can always choose a destabilizing factor v with positive rank and smaller slope. As $\Delta(v)$ is less than $\Delta(w)$, there are finitely many candidates v . By checking the positions of v and $v - w$ on the $\{1, \frac{\text{ch}_1}{\text{ch}_0}, \frac{\text{ch}_2}{\text{ch}_0}\}$ -plane, which are purely numerical data, Theorem 3.16 determines whether $L_{w\sigma}$ is an actual wall induced by this pair. The idea of the proof is very similar to that of the last wall, we first show there are stable objects on the wall with characters v and $w - v$ by Theorem 3.14. We then argue that the ext^1 of the stable objects is greater than 0 by Lemma 3.4, and finally claim that curves must be contracted from the σ_+ -side wall-crossing.

To state the criteria for actual walls, we first need to introduce the following definition.

Definition 3.15. For a Chern character w with $\text{ch}_0(w) \geq 0$ and an exceptional character e , we define the triangle TR_{we} to be the triangle region bounded by lines L_{we} , $L_{e^l e^+}$ and $L_{e^+ e^r}$ in the $\{1, \frac{\text{ch}_1}{\text{ch}_0}, \frac{\text{ch}_2}{\text{ch}_0}\}$ -plane.

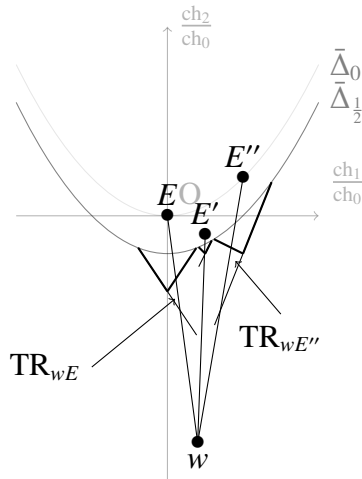


Figure: Definition of TR_{wE} .

Now we can state the main theorem on actual walls. The regions TR_{wE} will be used to detect the non-emptiness of moduli spaces of stable object of any ‘sub-character’, as will be explained in the proof of the theorem.

Theorem 3.16. *Let $w \in K(\mathbf{P}^2)$ be a Chern character outside the Le Potier cone with $\text{ch}_0(w) \geq 0$. For any stability condition $\sigma_{s,q}$ in $\bar{\Delta}_{<0}$ between the wall L_w^{last} and the vertical ray L_{w+} , the wall $L_{\sigma w}$ is an actual wall for w if and only if there exists a Chern character $v \in K(\mathbf{P}^2)$ on the line segment $l_{\sigma w}$ such that:*

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- $\text{ch}_0(v) > 0$ and $\frac{\text{ch}_1(v)}{\text{ch}_0(v)} < \frac{\text{ch}_1(w)}{\text{ch}_0(w)}$;
- the characters v and $w - v$ are either exceptional or not inside the Le Potier cone and both of them are not in TR_{wE} for any exceptional bundle E .

Remark 3.17. 1. For given characters w and v , one only needs to check whether v or $w - v$ are in TR_{wE} for at most two particular exceptional bundles. Suppose the intersection points $L_{\sigma w} \cap \bar{\Delta}_{\frac{1}{2}}$ fall between the segment between e_i^r and e_i^l for some exceptional character e_1 and e_2 , then one only needs to check the triangles TR_{wE_i} .

2. By the term ‘in TR_{wE} ’, strictly speaking, we mean that ‘in the closure of TR_{wE} but not on the line $L_{e^+e^l}$ when E is not to the right of E_w or not on the line $L_{e^+e^r}$ when E is to the left of E_w ’.

Proof. The first step is to show that when v (or $w - v$) is not inside the Le Potier cone, the condition ‘ v (or $w - v$) is not in TR_{wE} for any exceptional E ’ is equivalent to ‘ $\mathfrak{M}_{\sigma}^s(v)$ (or $\mathfrak{M}_{\sigma}^s(w - v)$) is non-empty for σ in $\bar{\Delta}_{<0}$ on the line L_{vw} ’.

The ‘ \Leftarrow ’ direction is easy to check: Suppose v is in TR_{wE} for some E , then E must be E_w or to the right of E_w . This implies $L_{\sigma w}$ intersects $L_{e^+e^r}$. The character v is in \mathfrak{R}_E and of Case 1 in Definition 3.11. By Proposition 1.27, $\mathfrak{M}_{\sigma}^s(v)$ is empty. The $w - v$ part is proved in a similar way.

For the ‘ \Rightarrow ’ direction, let f_1 and f_2 be the intersection points of the line L_{vw} and the parabola $\bar{\Delta}_{\frac{1}{2}}$. Suppose that f_1 has larger $\frac{\text{ch}_1}{\text{ch}_0}$, and f_1 lies on the segment between e^l and e^r for some exceptional bundle E by Theorem 4.1 in [CHW]. Since v is below L_{ee^r} , E_v is either E or to the left of E .

Three different cases may happen:

1. If v is above $L_{e^+e^l}$, then $E_v = E$. Since v is not in TR_{wE} , v is above L_{Ew} . This implies E is below L_{vw} , and $L_{v\sigma}$ is between L_v^{last} and $L_{v\pm}$.
2. If $E \neq E_w$ and v is not above $L_{e^+e^l}$, then w is below the line $L_{e(-3)e^l}$ and $E(-3)$ is below L_{vw} . Hence, $L_{v\sigma}$ is between $L_{vE(-3)}$ and $L_{v\pm}$;
3. If $E = E_w$ and v is not above the line $L_{e^l e^+}$, then by Remark 3.17, v is above $L_w^{\text{last}} = L_{E(-3)w}$. w is below $L_v^{\text{last}} = L_{E(-3)v}$, therefore, $L_{wv\sigma}$ is between L_v^{last} and $L_{v\pm}$.

In either case, $L_{v\sigma}$ is between L_v^{last} and $L_{v\pm}$. It follows from Theorem 3.14 that $\mathfrak{M}_{\sigma}^s(v)$ is non-empty for any $\sigma \in L_{v\sigma} \cap \bar{\Delta}_{\leq 0}$.

Write u for $w - v$. When $\text{ch}_0(u) \geq 0$, by Lemma 3.13 and the similar argument as for v , $\mathfrak{M}_{\sigma}^{ss}(u)$ is non-empty for any $\sigma \in L_{vw} \cap \bar{\Delta}_{\leq 0}$. If $\text{ch}_0(u) < 0$, let E be the exceptional bundle such that f_2 lies on the segment of $\bar{\Delta}_{\frac{1}{2}}$ between $x = e^l$ and $x = e^r$. By a similar argument as for v , when u is above $L_{e^+e^l}$, L_{uw} is between $L_u^{\text{right-last}} = L_{uE}$ and $L_{i\pm}$. When u is not above $L_{e^+e^l}$, L_{uw} is between $L_{uE(3)}$ and $L_{u\pm}$. By Theorem 3.14, $\mathfrak{M}_{\sigma}^{ss}(u)$ is non-empty for any $\sigma \in L_{vw} \cap \bar{\Delta}_{\leq 0}$.

The second step is to prove the ‘only if’ direction in the statement, which follows from the claim in the first step. If $L_{\sigma w}$ is an actual wall, an object F with character w is destabilized by a stable object with character v such that $\text{ch}_0(v) > 0$ and $\frac{\text{ch}_1(v)}{\text{ch}_0(v)} < \frac{\text{ch}_1(w)}{\text{ch}_0(w)}$.

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By Corollary 1.30, v is exceptional or outside Cone_{LP} . By the previous discussion, since $\mathfrak{M}_\sigma^s(v)$ is not empty, v is not in any TR_{wE} . For the character $w - v$, since $\mathfrak{M}_\sigma^{ss}(w - v)$ is non-empty, we only need to consider the case when all semistable objects are strictly semistable. Since $\mathfrak{M}_\sigma^{ss}(w - v)$ is non-empty, by Proposition 2.15, Proposition 2.18 and Theorem 3.14, $\mathfrak{M}_\sigma^s(w - v) = \emptyset$ if and only if $L_{w\sigma}$ is $L_{w-v}^{\text{right-last}}$ or L_{w-v}^{last} . The second case is not possible by Lemma 3.13. We may assume $\text{ch}_0(v) > \text{ch}_0(w)$. $v - w$ is of Case 1 in Definition 3.11 since otherwise $v - w$ is not in $\text{TR}_{E_{v-w}^{(\text{rhs})} w}$ and $\mathfrak{M}_\sigma^s(w - v)$ is not empty. Write $E_{v-w}^{(\text{rhs})}$ as E , we let

$$v' := v - \chi(v - w, E) \cdot e,$$

then $w - v' = w - v + \chi(v - w, E) \cdot e$. Since $\chi(w - v', E) = 0$, $w - v'$ is the intersection point of $L_{w\sigma}$ and $L_{e^+e'}$ and is not in TR_{wE} . By the same argument as in Case 3.II.2 of the proof of Theorem 3.14, $\text{ch}_0(v') > 0$, $\frac{\text{ch}_1}{\text{ch}_0}(v') < \frac{\text{ch}_1}{\text{ch}_0}(w)$ and $\mathfrak{M}_\sigma^s(v')$ is non-empty. Therefore, the pair v' and $w - v'$ satisfies the requirements in the statement.

The last step is to prove the ‘if’ direction in the statement. Similar to the proof for the last wall, objects with characters v and $u = w - v$ do not always have non-trivial extensions. We need to build Chern characters u' and v' on the line L_{vw} , such that:

1. $w = v' + u'$;
2. $\mathfrak{M}_\sigma^s(v')$ and $\mathfrak{M}_\sigma^s(u')$ are non-empty for $\sigma \in L_{vw} \cap \bar{\Delta}_{<0}$.
3. $\mathfrak{M}_{\sigma_+}^s(v', u') \rightarrow \mathfrak{M}_\sigma^{ss}(w)$ contracts curves.

Four cases may happen for u and v :

i) v and u are not proportional to any exceptional characters, in other words, they are outside Cone_{LP} . Since they are outside the triangles TR_{wE} , $\mathfrak{M}_\sigma^s(v)$ and $\mathfrak{M}_\sigma^s(u)$ are non-empty. The characters v and u satisfy the conditions in Lemma 3.4 due to the same argument as Case 1 in Theorem 3.14. This implies $\chi(v, u) < 0$. By the first property of Lemma 2.17 and the same computation as in Proposition 2.18, $\chi(u, v) - \chi(v, u) \leq -3$. Therefore, by Lemma 2.9, for any σ -stable objects F and G in $\mathfrak{M}_\sigma^s(v)$ and $\mathfrak{M}_\sigma^s(u)$, $\text{ext}^1(G, F) \geq 3$. By Lemma 3.1, $\mathfrak{M}_{\sigma_+}^s(F, G) \rightarrow \mathfrak{M}_\sigma^{ss}(w)$ contracts curves.

ii) v is proportional to the character e of some exceptional bundle E , but u is not proportional to any exceptional characters. Write $v = ne$ for some integer $n \geq 1$. When $\text{ch}_0(w) \geq \text{ch}_0(e)$, the character $u' = u + (n - 1)e$ is to the right of w . Therefore u' is not in TR_{wE} and $\mathfrak{M}_\sigma^s(u')$ is non-empty. $v' = w - u' = e$ and $\mathfrak{M}_\sigma^s(e)$ is non-empty.

In the case $\text{ch}_0(e) > \text{ch}_0(w)$, we have:

$$\frac{\text{ch}_1(u)}{\text{ch}_0(u)} = \frac{\text{ch}_1(w - ne)}{\text{ch}_0(w - ne)} > \frac{\text{ch}_1(w - e)}{\text{ch}_0(w - e)} = \frac{\text{ch}_1(u')}{\text{ch}_0(u')}.$$

As u is outside Cone_{LP} and on the different component of $L_{w\sigma} \cap \bar{\Delta}_{>0}$ than that of w , u' is also outside Cone_{LP} and not in any TR_{wE} . We may still let v' be e .

As L_{ew} is not the last wall L_w^{last} , $\chi(e, w) \leq 0$. We have $\chi(e, u') \leq -1$. By the same argument as in i), we have $\text{ext}^1(G, E) \geq 3$ for any object G in $\mathfrak{M}_\sigma^s(u')$. Therefore, $\mathfrak{M}_{\sigma_+}^s(E, G) \rightarrow \mathfrak{M}_\sigma^{ss}(w)$ contracts curves.

iii) u is proportional to the character e of some exceptional bundle E , but v is not proportional to any exceptional characters. As u is not on the line segment $l_{\sigma w}$, it has negative ch_0 . Suppose $u = -ne$ and we may let $v' = w + e$ and $u' = -e$ in the similar way as in ii). By the same argument on the slope of v and v' , v' is outside any triangle are TR_{wE} . As L_{we} is not the last wall L_w^{last} , we have $\chi(w, e) \geq 0$. Therefore $\chi(v', u') \leq -1$. By the same argument as in i), we have $\text{ext}^1(E, F) \geq 3$ for any object F in $\mathfrak{M}_{\sigma}^s(v')$. Therefore, $\mathfrak{M}_{\sigma_+}^s(E, G) \rightarrow \mathfrak{M}_{\sigma}^{ss}(w)$ contracts curves.

iv) The Chern characters u and v are proportional to the characters e_1 and e_2 of exceptional bundles E_1 and E_2 respectively. Write $u = -n_1 e_1$ and $v = n_2 e_2$. Since L_{vw} is not L_w^{last} , $\chi(e_2, w) \leq 0$. Therefore,

$$n_2 \leq n_1 \chi(e_2, e_1) = n_1 \text{ext}^2(E_2, E_1) = n_1 \text{hom}(E_1, E_2(-3)) < n_1 \text{hom}(E_1, E_2).$$

As a consequence, we see that

$$\dim \mathfrak{M}_{\sigma_-}^s(E_1^{\oplus n_1}[1], E_2^{\oplus n_2}) = \dim \text{Kr}_{\text{hom}(E_1, E_2)}(n_1, n_2) = n_1 n_2 \text{hom}(E_1, E_2) - n_1^2 - n_2^2 + 1 \geq 2.$$

Here $\text{Kr}_{\text{hom}(E_1, E_2)}(n_1, n_2)$ is the Kronecker model, in other words, it is the representations space of $\text{Hom}(\mathbb{C}^{n_2}, \mathbb{C}^{n_1})^{\oplus \text{hom}(E_1, E_2)}$ quotient by the natural group action of $GL(n_1) \times GL(n_2)/\mathbb{C}^*$.

In all cases, L_{vw} is an actual wall for w . □

Now the following corollary follows easily:

Corollary 3.18 (Lower rank walls). *Let w be a character with $\text{ch}_0(w) \geq 0$. For any character v with $0 < \text{ch}_0(v) \leq \text{ch}_0(w)$, suppose that v is between the wall L_w^{last} and the vertical ray L_{w+} , outside the Le Potier cone Cone_{LP} , and not in TR_{wE} for any exceptional bundle E . Then L_{vw} is an actual wall.*

4 Applications: the ample cone and the movable cone

In this section, we work out several applications of our criteria on actual walls. We compute the boundary of the movable cone in Section 4.1 and the boundary of the nef cone in Section 4.2. In Section 4.3, we compute all the actual walls of moduli space of stable sheaves of character $(4, 0, -15)$, as an example of how to apply the machinery in this paper to a concrete situation.

4.1 Movable cone

Let $w \in K(\mathbf{P}^2)$ be a character with $\text{ch}_0(w) \geq 0$ not inside Cone_{LP} . It has been revealed in [CHW] that when σ is in the ‘last’ chamber above L_w^{last} , the birational model $\mathfrak{M}_{\sigma}^s(w)$ is of Picard number 1 if and only if w is right orthogonal to E_w . In other words, the movable cone boundary on the primary side is not the same as the effective cone boundary if and only if $\chi(E_w, w) = 0$. In this section, we determine the boundary of the movable cone in this case.

4.1 Movable cone

Let $(E_\alpha, E_\beta, E_\gamma)$ be a triple of exceptional bundles corresponding to dyadic numbers $\frac{p-1}{2^n}, \frac{p+1}{2^n}, \frac{p}{2^n}$, respectively. The following property is well-known, the reader is referred to [GR].

Lemma 4.1. *For the triple $(E_\alpha, E_\beta, E_\gamma)$, we have*

$$\chi(E_\alpha, E_\gamma) = \text{hom}(E_\alpha, E_\gamma) = 3\text{ch}_0(E_\beta),$$

$$\chi(E_\gamma, E_\beta) = \text{hom}(E_\gamma, E_\beta) = 3\text{ch}_0(E_\alpha),$$

$$\text{hom}(E_\alpha, E_\gamma) \cdot \text{hom}(E_\gamma, E_\beta) - \text{hom}(E_\alpha, E_\beta) = 3\text{ch}_0(E_\gamma).$$

$$\text{hom}(E_\beta(-3), E_\alpha) \text{hom}(E_\alpha, E_\gamma) - \text{hom}(E_\beta(-3), E_\gamma) = 3\text{ch}_0(E_\alpha)$$

For any exceptional $E_{(\frac{t}{2q})}$ such that $\frac{p-1}{2^n} < \frac{t}{2q} < \frac{p}{2^n}$, we have $\text{ch}_0(E_{(\frac{t}{2q})}) < \text{ch}_0(E_\gamma)$.

The following observation is from the proof for Proposition 3.14. It will be used in the next theorem.

Lemma 4.2. *Let $w \in K(\mathbf{P}^2)$ be a Chern character with outside Cone_{LP} . Let e be an exceptional character such that in the $\{1, \frac{\text{ch}_1}{\text{ch}_0}, \frac{\text{ch}_2}{\text{ch}_0}\}$ -plane, w is in the area between two parallel lines $L_{ee(3)}$ and $L_{e^+e^+}$. Then we have $|\text{ch}_0(w)| > \text{ch}_0(E)$.*

Let w be a primitive character outside Cone_{LP} with $\text{ch}_0(w) \geq 0$. Assume that w is right orthogonal to the exceptional bundle $E_w = E_\gamma$, and consider the triple $(E_\alpha, E_\beta, E_\gamma)$ corresponding to dyadic numbers $\frac{p-1}{2^n}, \frac{p+1}{2^n}, \frac{p}{2^n}$. The character w can be uniquely written as $n_2 e_\alpha - n_1 e_{\beta-3}$ for positive numbers n_1, n_2 .

Theorem 4.3. *Adopt the notations as above, we may define a character P based on two different cases:*

i) $P := e_\gamma - (3\text{ch}_0(E_\beta) - n_2)e_\alpha$, if $1 \leq n_2 < 3\text{ch}_0(E_\beta)$;

ii) $P := e_\gamma$, if $n_2 \geq 3\text{ch}_0(E_\beta)$.

On the wall L_{Pw} , a divisor of $\mathfrak{M}_{L_{Pw+}}^s(w)$ is contracted.

Proof. In order to apply Theorem 3.16, we first show that in case i), P is not inside Cone_{LP} , and is outside TR_{wE} for any exceptional bundle E . Since $\chi(e_\beta, P) = 0$, P is on the line $L_{e_\beta^+ e_\beta^l}$. By Lemma 4.1,

$$\begin{aligned} \chi(P, P) &= 1 + (3\text{ch}_0(E_\beta) - n_2)^2 - (3\text{ch}_0(E_\beta) - n_2)\chi(E_\alpha, E_\gamma) \\ &= 1 - n_2(3\text{ch}_0(E_\beta) - n_2) \leq 0 \end{aligned}$$

Since P is on the line $L_{e_\beta^+ e_\beta^l}$, it is not inside Cone_{LP} and is outside TR_{wE} for any exceptional bundle E .

We next show that $w - P$ is outside TR_{wE} for any exceptional bundle E . By Lemma 3.13, we only need to treat the case when $\text{ch}_0(w - P) < 0$. We are going to prove that for

4.1 Movable cone

any exceptional bundle E to the left of $E_\gamma(-3)$, if E is above L_{Pw} , then $\chi(P - w, E) \leq 0$. This will imply that $w - P$ is not in TR_{wE} .

In case *i*),

$$P - w = n_1 e_\beta(-3) - (3\text{ch}_0(E_\beta) \cdot e_\alpha - e_\gamma).$$

By Lemma 4.1 and Serre duality,

$$\chi(3\text{ch}_0(E_\beta)e_\alpha - e_\gamma, e_\alpha(-3)) = 3\text{ch}_0(E_\beta) - \text{hom}(e_\alpha, e_\gamma) = 0.$$

Since $\chi(e_\beta(-3), e_\alpha(-3))$ is also 0, the point $P - w$ is on the line $L_{e_\alpha(-3)^+ e_\alpha(-3)^r}$. $\text{ch}_0(w) \geq 0$ implies $n_2 \text{ch}_0(e_\alpha) \geq n_1 \text{ch}_0(e_\beta)$. Hence

$$n_1 \leq \frac{\text{ch}_0(e_\alpha)}{\text{ch}_0(e_\beta)} \cdot n_2 < 3\text{ch}_0(e_\alpha).$$

By the fourth equation in Lemma 4.1,

$$\begin{aligned} \chi(P - w, P - w) &= n_1^2 + 1 - n_1 \chi(e_\beta(-3), 3\text{ch}_0(E_\beta) \cdot e_\alpha - e_\gamma) \\ &= n_1^2 + 1 - 3\text{ch}_0(e_\alpha) \cdot n_1 < 0. \end{aligned}$$

Combining with the result that $P - w$ is on the line $L_{e_\alpha(-3)^+ e_\alpha(-3)^r}$, we know that $P - w$ is not above the curve C_{LP} , and for any exceptional bundle E to the left of $e_\alpha(-3)$,

$$\chi(P - w, E) \leq 0.$$

On the other hand, the line segment $l_{(P-w)P}$ is above the line $L_{(P-w)e_\gamma}$, hence above the line segment $l_{e_\alpha(-3)^r e_\gamma}$. Since $l_{e_\alpha(-3)^r e_\gamma}$ is above any exceptional characters between the vertical rays $L_{e_\alpha(-3)^\pm}$ and $L_{e_\gamma^\pm}$, the character $P - w$ is not in the triangle TR_{wE} for any such exceptional bundle E .

In case *ii*), the character $P - w$ can be rewritten as follows:

$$\begin{aligned} P - w &= e_\gamma - w \\ &= n_1 e_{\beta-3} - (n_2 - 3\text{ch}_0(E_\beta))e_\alpha - (3\text{ch}_0(E_\beta) \cdot e_\alpha - e_\gamma) \\ &= (n_2 - 3\text{ch}_0(E_\beta)) \left(\frac{\text{ch}_0(E_\alpha)}{\text{ch}_0(E_\beta)} e_{\beta-3} - e_\alpha \right) \\ &\quad + \left(n_1 - n_2 \frac{\text{ch}_0(E_\alpha)}{\text{ch}_0(E_\beta)} + 3\text{ch}_0(E_\alpha) \right) e_{\beta-3} - (3\text{ch}_0(E_\beta)e_\alpha - e_\gamma). \end{aligned}$$

Note that the character $\frac{\text{ch}_0(E_\alpha)}{\text{ch}_0(E_\beta)} e_{\beta-3} - e_\alpha$ in the first term is proportional to $e_\gamma(-3) - e_\gamma$ by a positive scalar, and the coefficient $n_2 - 3\text{ch}_0(E_\beta)$ is non-negative. We denote the rest term as

$$v' := \left(n_1 - n_2 \frac{\text{ch}_0(E_\alpha)}{\text{ch}_0(E_\beta)} + 3\text{ch}_0(E_\alpha) \right) e_{\beta-3} - (3\text{ch}_0(E_\beta)e_\alpha - e_\gamma).$$

By Lemma 4.1 and the assumption, $\text{ch}_0(v') = \text{ch}_0(P - w) > 0$. In particular, $n_1 - n_2 \frac{\text{ch}_0(E_\alpha)}{\text{ch}_0(E_\beta)} + 3\text{ch}_0(E_\alpha) > 0$. Since $\text{ch}_0(w) > 0$, we have the inequality:

$$n_1 - n_2 \frac{\text{ch}_0(E_\alpha)}{\text{ch}_0(E_\beta)} + 3\text{ch}_0(E_\alpha) < 3\text{ch}_0(E_\alpha).$$

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Due to a similar computation as in case *i*),

$$\begin{aligned}
\chi(v', v') &< \left(n_1 - n_2 \frac{\text{ch}_0(E_\alpha)}{\text{ch}_0(E_\beta)} + 3\text{ch}_0(E_\alpha) \right)^2 + 1 - \\
&\quad \left(n_1 - n_2 \frac{\text{ch}_0(E_\alpha)}{\text{ch}_0(E_\beta)} + 3\text{ch}_0(E_\alpha) \right) \chi(e_{\beta-3}, 3\text{ch}_0(E_\beta)e_\alpha - e_\gamma) \\
&< (3\text{ch}_0(E_\alpha)) \left(\left(n_1 - n_2 \frac{\text{ch}_0(E_\alpha)}{\text{ch}_0(E_\beta)} + 3\text{ch}_0(E_\alpha) \right) - 3\text{ch}_0(E_\alpha) \right) + 1 \\
&< 0.
\end{aligned}$$

Note that v' is on the line $L_{e_\alpha(-3)^+e_\alpha(-3)^r}$, v' is not above C_{LP} . Since v' is to the left of $E_\beta(-3)$, after moving along the direction $e_\gamma(-3) - e_\gamma$, $v' + a(e_\gamma(-3) - e_\gamma)$ is still not above C_{LP} or $L_{e_\alpha(-3)^+e_\alpha(-3)^r}$. Therefore, $P - w$ is not above those two curves. It is not in TR_{wE} for any E to the left of $E_\alpha(-3)$.

For any exceptional e between $L_{e_\alpha(-3)\pm}$ and $L_{e_\gamma(-3)\pm}$, by the assumption, $\text{ch}_0(P - w) \leq \text{ch}_0(e_\gamma) < \text{ch}_0(e)$. By Lemma 4.2, $P - w$ is not in the area between $L_{e^+e^r}$ and $L_{ee(3)}$. Since w is above $L_{ee(3)}$, $P - w$ is not in TR_{wE} for any exceptional E between $L_{e_\alpha(-3)\pm}$ and $L_{e_\gamma(-3)\pm}$.

The line segment $l_{(P-w)P}$ is above the character $e_\gamma(-3)$, hence above the line segment $l_{e_\gamma(-3)^re_\gamma}$. Since $l_{e_\gamma(-3)^re_\gamma}$ is above any exceptional characters between the vertical rays $L_{e_\alpha(-3)\pm}$ and $L_{e_\gamma\pm}$, the character $P - w$ is not in the triangle TR_{wE} for any such exceptional bundle E .

We finish the claim that $w - P$ is outside TR_{wE} for any exceptional bundle E . By Theorem 3.16, we know that L_{Pw} is an actual wall.

The last step is to show that a divisor of $\mathfrak{M}_{L_{Pw}^+}^s(w)$ is contracted at L_{Pw} . By Proposition 2.15 and Theorem 3.14, for $\sigma \in L_{Pw}$, $\dim \mathfrak{M}_{\sigma^+}^s(w - P) = 1 - \chi(P - w, P - w)$, and $\dim \mathfrak{M}_{\sigma^+}^s(P) = 1 - \chi(P, P)$. By the previous argument, they are both nonnegative.

By Lemma 2.9 and Lemma 3.1,

$$\begin{aligned}
\dim \mathfrak{M}_{\sigma^+}^s(w - P, P) &= \dim \mathfrak{M}_{\sigma^+}^s(w - P) + \dim \mathfrak{M}_{\sigma^+}^s(P) + \text{ext}^1(w - P, P) - 1 \\
&= 1 - \chi(w - P, w - P) - \chi(P, P) - \chi(w - P, P) \\
&= 1 - \chi(w, w) + \chi(P, w - P) \\
&= \dim \mathfrak{M}_{\sigma^+}^s(w) + \chi(P, w - P).
\end{aligned}$$

So it suffices to show that $\chi(P, w - P) = -1$. This is clear in case *ii*):

$$\chi(P, w - P) = \chi(e_\gamma, w - e_\gamma) = -\chi(e_\gamma, e_\gamma) = -1.$$

4.2 Nef cone

In case i),

$$\begin{aligned}
\chi(P, w - P) &= -\chi(P, P) + \chi(P, w) \\
&= -\chi(P, P) - \chi((3\text{ch}_0(E_\beta) - n_2)e_\alpha, w) \\
&= -\chi(P, P) - (3\text{ch}_0(E_\beta) - n_2) \cdot n_2 \chi(e_\alpha, e_\alpha) + (3\text{ch}_0(E_\beta) - n_2) \cdot n_1 \chi(e_\alpha, e_{\beta-3}) \\
&= -\chi(e_\gamma, e_\gamma) - (3\text{ch}_0(E_\beta) - n_2)^2 + (3\text{ch}_0(E_\beta) - n_2) (\chi(e_\alpha, e_\gamma) + \chi(e_\gamma, e_\alpha)) \\
&\quad - (3\text{ch}_0(E_\beta) - n_2)n_2 \\
&= -1 - (3\text{ch}_0(E_\beta) - n_2)^2 + (3\text{ch}_0(E_\beta) - n_2) \cdot 3\text{ch}_0(E_\beta) - (3\text{ch}_0(E_\beta) - n_2)n_2 \\
&= -1.
\end{aligned}$$

□

4.2 Nef cone

In this section we study the boundary of the nef cone of the moduli space $\mathfrak{M}_{GM}^{ss}(w)$. Due to Theorem 2.24, this is the first actual wall to the left of the vertical wall $L_{w\pm}$. We assume that the character w is primitive, $\text{ch}_0(w) > 0$ and $\frac{\text{ch}_1(w)}{\text{ch}_0(w)} \in (-1, 0]$. The following lemma gives an obvious bound for the boundary of the nef cone.

Lemma 4.4. *Suppose $\bar{\Delta}(w) \geq 2$, then $L_{O(-1)w}$ is an actual wall for w .*

Proof. By Corollary 3.18 and Theorem 3.14, we need to show that w is below the line $L_{O(-1)O(-1)^r}$.

The point $O(-1)^r$ is the intersection of $\bar{\Delta}_{\frac{1}{2}}$ and $L_{OO(-1)}$, so on the $\{1, \frac{\text{ch}_1}{\text{ch}_0}, \frac{\text{ch}_2}{\text{ch}_0}\}$ -plane, its coordinate $\left(1, \frac{\text{ch}_1}{\text{ch}_0}, \frac{\text{ch}_2}{\text{ch}_0}\right) = \left(1, \frac{1-\sqrt{5}}{2}, \frac{1-\sqrt{5}}{4}\right)$. Let P be the intersection point of $L_{O(-1)O(-1)^r}$ and $L_{O\pm}$. The function $\bar{\Delta}$ on the line segment $l_{O(-1)P}$ reaches its maximum at $P = (1, 0, -\frac{1+\sqrt{5}}{2})$, and $\bar{\Delta}_P = \frac{1+\sqrt{5}}{2} < 2$. So w is below the line $L_{O(-1)O(-1)^r}$. □

By the lemma, when $\frac{\text{ch}_1(w)}{\text{ch}_0(w)} \in (k-1, k]$ for some integer k and $\bar{\Delta}(w) \geq 2$, $L_{O(k-1)w}$ is an actual wall.

Lemma 4.5. *Suppose $\bar{\Delta}(w) \geq 10$, then the first lower rank wall L_{vw} with $\text{ch}_0(v) \leq \text{ch}_0(w)$ is given by the character v satisfying the following two conditions:*

- $\frac{\text{ch}_1(v)}{\text{ch}_0(v)}$ is the greatest rational number less than $\frac{\text{ch}_1(w)}{\text{ch}_0(w)}$ with $\text{ch}_0(v) \leq \text{ch}_0(w)$;
- Given the first condition, if $\text{ch}_1(v)$ is even (odd resp.), let $\text{ch}_2(v)$ be the greatest integer ($2\text{ch}_2(v)$ be the greatest odd integer resp.), such that the point v is either an exceptional character or not inside Cone_{LP} .

Proof. We may assume that $0 < \frac{\text{ch}_1(w)}{\text{ch}_0(w)} \leq 1$. Note that the slopes of $L_{e^+e^+}$ and $L_{e^+e^-}$ for any exceptional object with $\frac{\text{ch}_1(e)}{\text{ch}_0(e)} \in [-1, 0]$ are at least $-\frac{5}{2}$.

We first show that there is no actual wall between L_{vw} and $L_{w\pm}$. Suppose that there is a character v' with $\text{ch}_0(v') \leq \text{ch}_0(w)$ and $\frac{\text{ch}_1(v')}{\text{ch}_0(v')} < \frac{\text{ch}_1(w)}{\text{ch}_0(w)}$, such that $L_{v'w}$ is an actual wall

4.2 Nef cone

between L_{vw} and $L_{w\pm}$. By the previous lemma, we may assume that $\frac{\text{ch}_1(v')}{\text{ch}_0(v')} \geq 1$. Since v' is either an exceptional character or below C_{LP} , by the assumptions on v ,

$$\frac{\text{ch}_2(v)}{\text{ch}_0(v)} - \frac{\text{ch}_2(v')}{\text{ch}_0(v')} \geq -\frac{1}{\text{ch}_0(v)} + \frac{5}{2} \left(\frac{\text{ch}_1(v')}{\text{ch}_0(v')} - \frac{\text{ch}_1(v)}{\text{ch}_0(v)} \right) - \frac{1}{\text{ch}_0(v')^2}.$$

The coefficient $\frac{5}{2}$ of second term is with respect to the minimum slope of the Le Potier curve. The last term is for the case that v' is exceptional: $\frac{\text{ch}_2(e)}{\text{ch}_0} - \frac{\text{ch}_2(e^+)}{\text{ch}_0} = \frac{1}{\text{ch}_0(e)^2}$. This inequality holds since otherwise $v - (0, 0, 1)$ will be below C_{LP} with smaller $\text{ch}_2(v)$.

Write $d_\chi := -\frac{\text{ch}_1(v')}{\text{ch}_0(v')} + \frac{\text{ch}_1(v)}{\text{ch}_0(v)}$ for simplicity, then $\frac{1}{\text{ch}_0(v)\text{ch}_0(v')} \leq d_\chi \leq 1$. Since $L_{v'w}$ is between L_{vw} and $L_{w\pm}$, in other words, v is below $l_{v'w}$, we have the inequality:

$$\begin{aligned} \frac{\text{ch}_2(v')}{\text{ch}_0} - \frac{\text{ch}_2(w)}{\text{ch}_0} &\leq \left(\frac{\text{ch}_2(v')}{\text{ch}_0} - \frac{\text{ch}_2(v)}{\text{ch}_0} \right) \frac{\frac{\text{ch}_1(w)}{\text{ch}_0} - \frac{\text{ch}_1(v')}{\text{ch}_0}}{d_\chi} \\ &\leq \left(\frac{1}{\text{ch}_0(v)} + \frac{1}{\text{ch}_0(v')^2} + \frac{5}{2}d_\chi \right) \cdot \left(1 + \frac{1}{d_\chi} \cdot \left(\frac{\text{ch}_1(w)}{\text{ch}_0} - \frac{\text{ch}_1(v)}{\text{ch}_0} \right) \right) \\ &\leq 1 + 1 + \frac{5}{2} + \frac{1}{d_\chi} \cdot \left(\frac{\text{ch}_1(w)}{\text{ch}_0} - \frac{\text{ch}_1(v)}{\text{ch}_0} \right) \cdot \left(\frac{1}{\text{ch}_0(v)} + \frac{1}{\text{ch}_0(v')^2} + \frac{5}{2}d_\chi \right) \\ &\leq \frac{9}{2} + \frac{1}{\text{ch}_0(w)} \left(\frac{1}{d_\chi} \left(\frac{1}{\text{ch}_0(v)} + \frac{1}{\text{ch}_0(v')} \right) + \frac{5}{2} \right) \\ &\leq \frac{9}{2} + 1 + 1 + \frac{5}{2} = 9. \end{aligned}$$

Therefore

$$\begin{aligned} \bar{\Delta}_w &= \frac{\text{ch}_2(w)}{\text{ch}_0} + \frac{1}{2} \left(\frac{\text{ch}_2(w)}{\text{ch}_0} \right)^2 \leq -\frac{\text{ch}_2(w)}{\text{ch}_0} + \frac{1}{2} \left(\frac{\text{ch}_2(v')}{\text{ch}_0} \right)^2 \\ &= -\frac{\text{ch}_2(w)}{\text{ch}_0} + \frac{\text{ch}_2(v')}{\text{ch}_0} + \bar{\Delta}_{v'} < 9 + 1 = 10, \end{aligned}$$

which contradicts to our assumption.

We next show that L_{vw} is an actual wall. By Corollary 3.18, it suffices to prove that v is not in TR_{wE} for any exceptional bundle E such that $-1 \leq \frac{\text{ch}_1(E)}{\text{ch}_0} \leq 0$. Suppose that v is in TR_{wE} for an exceptional bundle E , then since $\bar{\Delta}_w \geq 10$, the slope of L_{wE} is less than -9 . The $\frac{\text{ch}_1}{\text{ch}_0}$ -width of TR_{wE} is less than

$$\frac{\text{the length of } l_{ee^+}}{9 - \frac{5}{2}} < \frac{1}{6\text{ch}_0(E)^2}.$$

Hence if v is in TR_{wE} , then

$$\frac{1}{\text{ch}_0(E)\text{ch}_0(v)} \leq \frac{\text{ch}_1(v)}{\text{ch}_0(v)} - \frac{\text{ch}_1(E)}{\text{ch}_0(E)} < \frac{1}{6\text{ch}_0(E)^2}.$$

In this way, $\text{ch}_0(w) \geq \text{ch}_0(v) > 6\text{ch}_0(E)$. In particular, $\text{ch}_0(E) \leq \text{ch}_0(w)$. Note that L_{wE} becomes a lower rank wall closer to between L_{vw} and $L_{w\pm}$. By Corollary 3.18, L_{wE} is an actual wall. But this is not possible by the proof in the first part. Therefore, v is not in TR_{wE} for any exceptional bundle E . \square

4.3 A concrete example

Now we can describe the boundary of the nef cone:

Theorem 4.6. *Let w be a primitive character with $\text{ch}_0(w) > 0$ and $\bar{\Delta}_w \geq 10$, the first actual wall for $\mathfrak{M}_{GM}^s(w)$ is given by L_{vw} , where v is the character defined in Lemma 4.5.*

Proof. We may assume that $\frac{\text{ch}_1(w)}{\text{ch}_0(w)} \in (-1, 0]$, and by Lemma 4.5, we only need to show that any higher rank actual wall is not between L_{vw} and $L_{w\pm}$. Let v' be a character satisfying the properties in Theorem 3.16 with $\text{ch}_0(v') = \text{ch}_0(w) + r$ for some positive integer r .

The slope of L_{vw} is less than

$$(\bar{\Delta}_w - 1) / \left(\frac{\text{ch}_1(w)}{\text{ch}_0(w)} - \frac{\text{ch}_1(v)}{\text{ch}_0(v)} \right) < -9\text{ch}_0(w).$$

So the left intersection point of $L_{vw} \cap \bar{\Delta}_0$ has $\frac{\text{ch}_1}{\text{ch}_0}$ -coordinate less than $-9\text{ch}_0(w)$. Since $v' - w$ is to the left of this point, we get the inequality

$$\frac{\text{ch}_1}{\text{ch}_0}(v' - w) < -9\text{ch}_0(w),$$

hence

$$r < \frac{1}{9} \cdot \frac{\text{ch}_1(w) - \text{ch}_1(v')}{\text{ch}_0(w)}.$$

By Lemma 4.2, we have $\frac{\text{ch}_1(v')}{\text{ch}_0(v')} > -1$, so

$$\text{ch}_1(v') > -\text{ch}_0(w) - r.$$

Therefore,

$$r < \frac{1}{9} \cdot \frac{\text{ch}_1(w) + \text{ch}_0(w) + r}{\text{ch}_0(w)} \leq \frac{1}{9} \cdot \frac{\text{ch}_0(w) + r}{\text{ch}_0(w)} \leq \frac{1}{9} + \frac{r}{9}.$$

This leads to the contradiction since $r < 1$ and cannot be a positive integer. \square

4.3 A concrete example

In this section, we apply the criteria for actual walls and compute the stable base locus walls on the primitive side for the moduli space of stable objects of character $w = (\text{ch}_0, \text{ch}_1, \text{ch}_2) = (4, 0, -15)$.

We first determine the last wall of w . The equation of $L_w^{\bar{\Delta}_1}$ is given by

$$\frac{\text{ch}_2}{\text{ch}_0} + \frac{3\sqrt{35}}{14} \frac{\text{ch}_1}{\text{ch}_0} + \frac{15}{4} = 0.$$

The $\frac{\text{ch}_1}{\text{ch}_0}$ coordinates of the intersection points $L_w^{\bar{\Delta}_1} \cap \bar{\Delta}_1$ are $-\frac{\sqrt{35}}{2} \pm \frac{3}{2}$. The larger one is approximately -1.458 and the intersection point falls in the segment between e^l and e^r , for the exceptional character e given by $E(\frac{3}{2})$, which is the cotangent bundle Ω .

By the Hirzebruch-Riemann-Roch formula in the proof for Lemma 2.17,

$$\chi(\Omega, w) = \chi\left((2, -3, \frac{3}{2}), (4, 0, -15)\right) = -30 + 6 + 18 + 8 = 2 > 0.$$

4.3 A concrete example

Therefore w is above the line L_{e^+} , and is of Case 1 in the Definition 3.11. The last wall of w is given by $L_{\Omega w}$ with equation:

$$\frac{\text{ch}_2}{\text{ch}_0} + 3\frac{\text{ch}_1}{\text{ch}_0} + \frac{15}{4} = 0.$$

We now look for all the lower rank walls. By Corollary 3.18, we only need determine all characters $v \in K(\mathbf{P}^2)$ such that

- $0 < \text{ch}_0(v) \leq \text{ch}_0(w) = 4$, $\frac{\text{ch}_1}{\text{ch}_0}(v) < \frac{\text{ch}_1}{\text{ch}_0}(w) = 0$;
- v is between L_{w^\pm} and L_w^{last} ;
- v is exceptional or not inside Cone_{LP} ;
- v is not in TR_{wE} for any exceptional character E .

When $\text{ch}_0(v)$ is 1, $\text{ch}_1(v)$ can only be -1 , v is either $(1, -1, \frac{1}{2})$ or $(1, -1, -\frac{1}{2})$.

When $\text{ch}_0(v)$ is 2, $\text{ch}_1(v)$ can be -1 or -2 , v is one of the characters as follows: $(2, -1, -\frac{1}{2})$; $(2, -1, -\frac{3}{2})$; $(2, -1, -\frac{5}{2})$; $(2, -1, -\frac{7}{2})$; $(2, -2, -1)$.

When $\text{ch}_0(v)$ is 3, $\text{ch}_1(v)$ can be -1 , -2 , -3 or -4 , v is one of the following characters:

- $(3, -1, -\frac{3}{2})$; $(3, -1, -\frac{5}{2})$; $(3, -1, -\frac{7}{2})$; ... $(3, -1, -\frac{15}{2})$;
- $(3, -2, -1)$; $(3, -2, -2)$; $(3, -2, -3)$; $(3, -2, -4)$; $(3, -2, -5)$;
- $(3, -3, -\frac{3}{2})$; $(3, -4, 1)$.

When $\text{ch}_0(v)$ is 4, we have $-5 \leq \text{ch}_1(v) \leq -1$, v is one of the following characters:

- $(4, -1, -\frac{5}{2})$; $(4, -1, -\frac{7}{2})$; ... $(4, -1, -\frac{23}{2})$;
- $(4, -2, -4)$; $(4, -2, -5)$; ... $(4, -2, -8)$;
- $(4, -3, -\frac{3}{2})$; $(4, -3, -\frac{5}{2})$; ... $(4, -3, -\frac{11}{2})$;
- $(4, -4, -2)$; $(4, -5, \frac{1}{2})$.

The nef boundary of $\mathfrak{M}_{GM}^s(w)$ is the wall $L_{w(4, -1, -\frac{5}{2})}$.

We now compute the characters that are contained in TR_{wE} for some exceptional E . By Lemma 4.2, we only need consider the exceptional bundles $\mathcal{O}(-1)$ and $\Omega(1)$. The equations for the three edges of $\text{TR}_{w\mathcal{O}(-1)}$ are:

$$\frac{\text{ch}_2}{\text{ch}_0} - \frac{1}{2}\frac{\text{ch}_1}{\text{ch}_0} = 0, \quad \frac{\text{ch}_2}{\text{ch}_0} + \frac{5}{2}\frac{\text{ch}_1}{\text{ch}_0} + 3 = 0, \quad \frac{\text{ch}_2}{\text{ch}_0} + \frac{17}{4}\frac{\text{ch}_1}{\text{ch}_0} + \frac{15}{4} = 0.$$

By a direct computation, characters $(3, -2, 3)$, $(4, -3, -\frac{5}{2})$, $(4, -3, -\frac{7}{2})$ are in $\text{TR}_{w\mathcal{O}(-1)}$. The equations for the three edges of $\text{TR}_{w\Omega(1)}$ are:

$$\frac{\text{ch}_2}{\text{ch}_0} - \frac{\text{ch}_1}{\text{ch}_0} = 0, \quad \frac{\text{ch}_2}{\text{ch}_0} + 2\frac{\text{ch}_1}{\text{ch}_0} + \frac{3}{2} = 0, \quad \frac{\text{ch}_2}{\text{ch}_0} + 7\frac{\text{ch}_1}{\text{ch}_0} + \frac{15}{4} = 0.$$

4.3 A concrete example

The coordinates of vertices are $(1, -\frac{1}{2}, -\frac{1}{2})$, $(1, -\frac{9}{20}, -\frac{3}{5})$, $(1, -\frac{15}{32}, -\frac{15}{32})$. Since for any v , $\frac{\text{ch}_1}{\text{ch}_0}(v)$ is not in $(-\frac{1}{2}, -\frac{9}{20})$, there is no v in $\text{TR}_{w\Omega(1)}$.

To find the higher rank walls, we first determine the bound for $\text{ch}_0(v)$. $L_w^{\text{last}} \cap \bar{\Delta}_{\leq 0} = \left\{ \left(1, -3 + \sqrt{\frac{3}{2}}, \frac{1}{2} \left(-3 + \sqrt{\frac{3}{2}} \right)^2 \right), \left(1, -3 - \sqrt{\frac{3}{2}}, \frac{1}{2} \left(-3 - \sqrt{\frac{3}{2}} \right)^2 \right) \right\}$. Let $v \in K(\mathbf{P}^2)$ be a character such that

- $\text{ch}_0(v) > \text{ch}_0(w) = 4$, $\frac{\text{ch}_1}{\text{ch}_0}(v) < \frac{\text{ch}_1}{\text{ch}_0}(w) = 0$;
- v is between $L_{w\pm}$ and L_w^{last} ;
- v and $v - u$ are exceptional or not inside Cone_{LP} ;
- v and $v - u$ are not in TR_{wE} for any exceptional character E .

Since v and $v - u$ are on the different components of $L_{vw} \cap \bar{\Delta}_{\geq 0}$, we have the inequalities:

$$\frac{\text{ch}_1}{\text{ch}_0}(v) \geq -3 + \sqrt{\frac{3}{2}}, \quad \frac{\text{ch}_1}{\text{ch}_0}(v - u) \leq -3 - \sqrt{\frac{3}{2}}.$$

Therefore,

$$\left(-3 + \sqrt{\frac{3}{2}} \right) \text{ch}_0(v) \leq \text{ch}_1(v) \leq \left(-3 - \sqrt{\frac{3}{2}} \right) (\text{ch}_0(v) - 4). \quad (3)$$

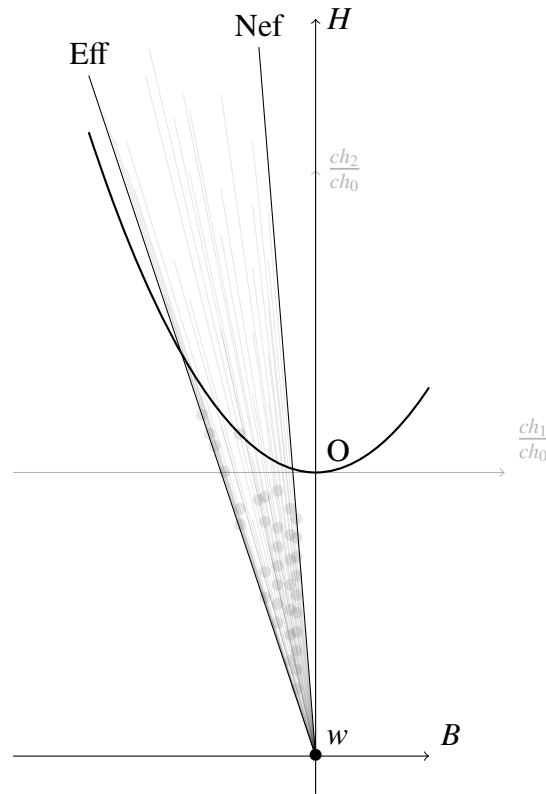
We get a bound for $\text{ch}_0(v)$: $\text{ch}_0(v) \leq 2 + 2\sqrt{6} < 7$. When $\text{ch}_0(v)$ is 6, by (3), $\text{ch}_1(v) \leq -2 \left(-3 - \sqrt{\frac{3}{2}} \right) < -8$. Therefore, $\frac{\text{ch}_1}{\text{ch}_0}(v) \leq -9 = \frac{\text{ch}_1}{\text{ch}_0}(E_w)$, which is not possible.

When $\text{ch}_0(v)$ is 5, by (3), $\text{ch}_1(v)$ can be -5 , -6 or -7 . v is one of the following characters:

$$(5, -5, -\frac{7}{2}); (5, -6, 0); (5, -7, \frac{5}{2}).$$

These characters v and $w - v$ are not contained in TR_{wE} for any exceptional E . Combining Theorem 2.24 and Theorem 3.16, we may draw the stable base locus decomposition walls in the divisor cone of $\mathfrak{M}_{GM}^s(w)$ as follows.

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The stable base locus decomposition of the effective cone of $\mathfrak{M}_{GM}^s(4, 0, -15)$

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